The Action of $SL_2(\mathbb{Z})$ on Origamis
A subset of the Quadratic Differentials

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January 26th, 2019
Outline

• Curves on Surfaces

• Origami

• $SL_2(\mathbb{Z})$ Action

• The Invariant and Our Results
Curves on Surfaces

We will consider curves that are:

- Simple, Closed, Essential, Oriented
- In Pairs: \((\lambda, \mu)\)
- Minimally Intersecting: \(\min(\lambda \cap \mu)\)
- Filling: \(\Sigma_g \setminus (\lambda \cup \mu) \cong D^2\)

Examples
Minimal Intersection

Lemma

Suppose $(\alpha, \beta)$ is a pair of curves which fill $\Sigma_g$, $g \geq 1$. Then $i(\alpha, \beta) = 2g - 1$.

Aougab-Haung 2013

Sketch of the proof:

- Let $i(\alpha, \beta)$ denote the intersection number of curves $(\alpha, \beta)$
- $\chi(X) = 2 - 2g = $ Vertices - Edges + Faces
- Since Vertices $= i(\alpha, \beta)$ and Edges $= 2i(\alpha, \beta)$,
- $2 - 2g = i(\alpha, \beta) - 2i(\alpha, \beta) + $ Faces
- $2 - 2g = -i(\alpha, \beta) + $ Faces $\geq -i(\alpha, \beta) + 1$
- $i(\alpha, \beta) \geq 2g - 1$.
- Since $\Sigma_g \setminus (\alpha \cup \beta)$ is a single disk, $i(\alpha, \beta) = 2g - 1$.

Relabel $n := 2g - 1.$
Surfaces

Examples of Surfaces

If we take the collection of all surfaces up to homeomorphism and quotient the set by conformal maps between surfaces, we get the Teichmüller space, $\mathcal{T}_g$.

The cotangent bundle of $\mathcal{T}_g$ is naturally identified with the space of quadratic differentials $\mathcal{QD}_g$.

Origamis are a subset of $\mathcal{QD}_g$. 
Origami

An *origami* is a surface obtained from gluing up the boundary of a region in $\mathbb{C}$ that is tiled by congruent squares.

Origamis can also be naturally identified as a branched cover of a square torus, where there is a single branch point.
SL₂(Z) Action

SL₂(ℝ) has a natural action on QDₕ:
SL₂(ℝ) × QDₕ → QDₕ

This action restricts to SL₂(Z) on the set of origamis:
SL₂(Z) × {Origamis} → {Origamis}

Take as a basis \((S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix})\) for SL₂(Z).

Then the action can be described in terms of the basis as follows:

\[ R \cdot (\tilde{\lambda}, \tilde{\mu}) = (\tilde{\mu}, \tilde{\lambda}^{-1}) \quad S \cdot (\tilde{\lambda}, \tilde{\mu}) = (\tilde{\lambda}, \tilde{\mu} \cdot \tilde{\lambda}^{-1}) \]

This action of SL₂(Z) partitions the set of all pairs \((\tilde{\lambda}, \tilde{\mu})\) into orbits.
**SL₂(ℤ) Action**

\[ R \cdot (\tilde{\lambda}, \tilde{\mu}) = (\tilde{\mu}, \tilde{\lambda}^{-1}) \quad \text{and} \quad S \cdot (\tilde{\lambda}, \tilde{\mu}) = (\tilde{\lambda}, \tilde{\mu} \cdot \tilde{\lambda}^{-1}) \]
Monodromy Explanation

Because an origami is a branched cover of $T^2$, there is an associated monodromy representation of the $\pi_1(T^2 \setminus \{pt\}) = F_2$.

The image of this representation is a subgroup of $S_n$, with $n$ the number of squares, called the *monodromy group*.

As a consequence of $n$ being odd, the monodromy group of the cover is $\leq A_n$.

It is known that the monodromy group serves as an invariant of the orbits under the $SL_2(\mathbb{Z})$ action on the set of origamis.
A Lower Bound

Main Question

For any $g$, how many orbits exist under this $SL_2(\mathbb{Z})$ action?

Under this invariant, we expect that $\forall \Sigma_g$ with $g > 4$, there are at least two orbits under the $SL_2(\mathbb{Z})$ action.

Our Results So Far

We have shown this to be true for $g = 5, 6, 7$. 
## Our Results

### Table 1: Orbit Data

<table>
<thead>
<tr>
<th>Genus</th>
<th>$n = 2g - 1$</th>
<th>Orbits</th>
<th>Monodromy Group(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g=3$</td>
<td>5</td>
<td>1</td>
<td>$A_5$</td>
</tr>
<tr>
<td>$g=4$</td>
<td>7</td>
<td>4</td>
<td>$A_7$</td>
</tr>
<tr>
<td>$g=5$</td>
<td>9</td>
<td>11</td>
<td>$A_9, L_2(8), L_2(8) \rtimes \mathbb{Z}_3$</td>
</tr>
<tr>
<td>$g=6$</td>
<td>11</td>
<td>$\geq 2$</td>
<td>$A_{11}, M_{11}$</td>
</tr>
<tr>
<td>$g=7$</td>
<td>13</td>
<td>$\geq 2$</td>
<td>$A_{13}, SL_3(3)$</td>
</tr>
</tbody>
</table>
Acknowledgements

Thanks to:

21st Annual NCUWM, 2019

University of California, Riverside

ICERM and my collaborators

National Science Foundation
References


