The Action of $SL_2(\mathbb{Z})$ on Origamis

A subset of the Quadratic Differentials

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Outline

Curves on Surfaces

We will consider curves that are:

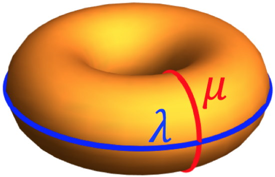
Simple, Closed, Essential, Oriented

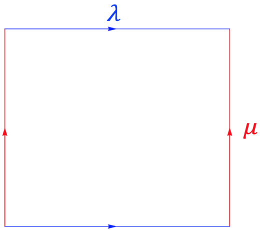
In Pairs: (λ, μ)

Minimally Intersecting: min$(\lambda \setminus \mu)$

Filling: Σ_g n $(\lambda$ [$\mu)$ ' D^2

Examples





Minimal Intersection

Lemma

Suppose (α, β) is a pair of curves which fill $\Sigma_g, g \quad 1$. Then $i(\alpha, \beta) = 2g \quad 1$. Aougab-Haung 2013

Sketch of the proof:

Let $i(\alpha, \beta)$ denote the intersection number of curves (α, β)

$\chi(X) = 2 \quad 2g =$ Vertices - Edges + Faces

Since Vertices $= i(\alpha, \beta)$ and Edges $= 2i(\alpha, \beta)$,

$2 \quad 2g = i(\alpha, \beta) \quad 2i(\alpha, \beta) +$ Faces

$2 \quad 2g = \quad i(\alpha, \beta) +$ Faces $\qquad i(\alpha, \beta) + 1$

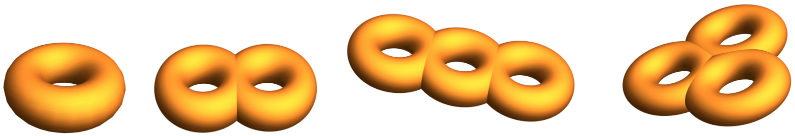
$i(\alpha, \beta) \quad 2g \quad 1$.

Since $\Sigma_g \, n \, (\alpha \; [\; \beta)$ is a single disk, $i(\alpha, \beta) = 2g \quad 1$.

Relabel $n := 2g \quad 1$.

Surfaces

Examples of Surfaces



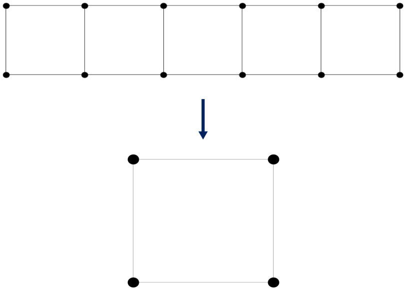
If we take the collection of all surfaces up to homeomorphism and quotient the set by conformal maps between surfaces, we get the Teichmüller space, T_g.

The cotangent bundle of T_g is naturally identified with the space of quadratic differentials QD_g.

Origamis are a subset of QD_g.

Origami

An *origami* is a surface obtained from gluing up the boundary of a region in $\mathbb{C}$ that is tiled by congruent squares.



Origamis can also be naturally identified as a branched cover of a square torus, where there is a single branch point.

$SL_2(\mathbb{Z})$ Action

$SL_2(\mathbb{R})$ has a natural action on QD_g:
$SL_2(\mathbb{R}) \quad QD_g \;\; ! \;\; QD_g$

This action restricts to $SL_2(\mathbb{Z})$ on the set of origamis:
$SL_2(\mathbb{Z}) \quad fOrigamisg \;\; ! \;\; fOrigamisg$

Take as a basis ($S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 0 & \;\; 1 \\ 1 & 0 \end{bmatrix}$) for $SL_2(\mathbb{Z})$.

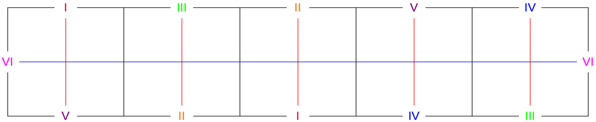
Then the action can be described in terms of the basis as follows:

$$R \quad (\tilde{\lambda}, \tilde{\mu}) = (\tilde{\mu}, \tilde{\lambda} \;\; ^1) \qquad\qquad S \quad (\tilde{\lambda}, \tilde{\mu}) = (\tilde{\lambda}, \tilde{\mu} \quad \tilde{\lambda} \;\; ^1)$$

This action of $SL_2(\mathbb{Z})$ partitions the set of all pairs $(\tilde{\lambda}, \tilde{\mu})$ into *orbits*.

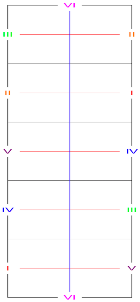
$SL_2(\mathbb{Z})$ Action

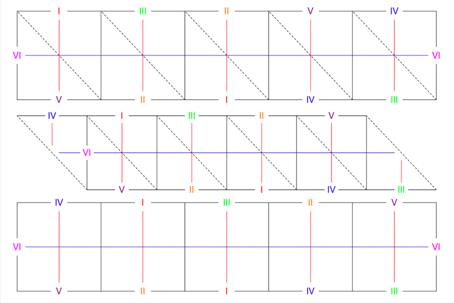
Mifpo: {(12345),(13254)}



$$R\ \ (\tilde{\lambda}, \tilde{\mu}) = (\tilde{\mu}, \tilde{\lambda}\ \ ^{1})$$

$$S\ \ (\tilde{\lambda}, \tilde{\mu}) = (\tilde{\lambda}, \tilde{\mu}\ \ \tilde{\lambda}\ \ ^{1})$$





Monodromy Explanation

Because an origami is a branched cover of T^2, there is an associated monodromy representation of the $\pi_1(T^2 n fptg) = F_2$.

The image of this representation is a subgroup of S_n, with n the number of squares, called the *monodromy group*.

As a consequence of n being odd, the monodromy group of the cover is A_n.

It is known that the monodromy group serves as an invariant of the orbits under the $SL_2(\mathbb{Z})$ action on the set of origamis.

A Lower Bound

Main Question

For any g, how many orbits exist under this $SL_2(\mathbb{Z})$ action?

Under this invariant, we expect that $8\Sigma_g$ with $g > 4$, there are at least two orbits under the $SL_2(\mathbb{Z})$ action.

Our Results So Far

We have shown this to be true for $g = 5, 6, 7$.

Our Results

Table 1: Orbit Data

Genus	$n = 2g$ 1	**Orbits**	**Monodromy Group(s)**
g=3	5	1	A_5
g=4	7	4	A_7
g=5	9	11	A_9, $L_2(8)$, $L_2(8) \rtimes \mathbb{Z}_3$
g=6	11	2	A_{11}, M_{11}
g=7	13	2	A_{13}, $SL_3(3)$

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