Vertex-Minimal Planar Graphs with a Prescribed Automorphism Group

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Automorphism Group of a Graph

Definition

A **permutation** of a set S is a bijection from S to itself.

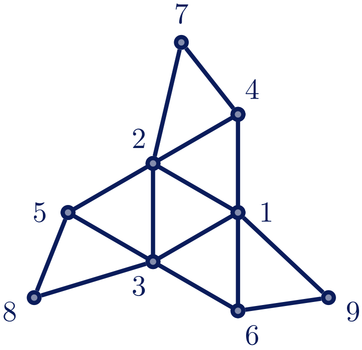
Definition

Let Γ be a graph. The **automorphism group** of Γ, denoted $\text{Aut}\,\Gamma$, is the set of adjacency preserving permutations of the vertices of Γ.

Example

Note that $\mathbb{Z}_3 \cong \{1,2,3\} \cong \langle (1,2,3)(4,5,6)(7,8,9) \rangle$. In this case, we have that

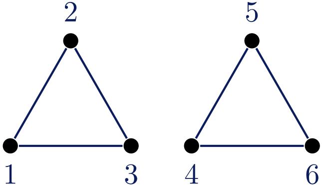
$\operatorname{Aut}\Gamma = \{(1), (1,2,3)(4,5,6)(7,8,9), (1,3,2)(4,6,5)(7,9,8)\}$.



Connected Components

Definition

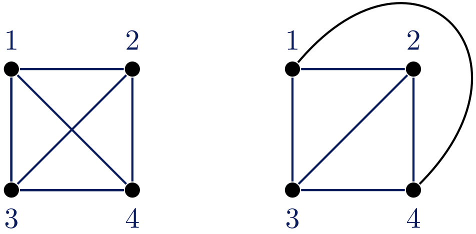
A **connected component** is a subgraph in which any two vertices are connected to each other by paths, and that is connected to no additional vertices in the supergraph.



Planar Graphs

Definition

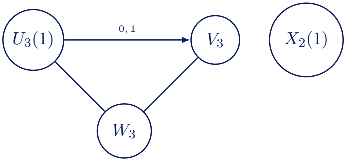
A graph is **planar** if it can be draw so that no edges intersect.



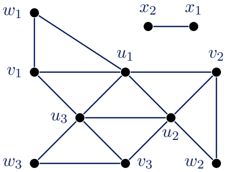
K_4 graph and a planar embedding.

F^*-diagrams

We can use F^*-diagrams to depict graphs that are too complicated to draw explicitly.



(a) F^*-diagram of Γ



(b) Depiction of Γ

$\alpha(G)$

Definition

For a finite group G, let $\boldsymbol{\alpha(G)}$ denote the minimum number of vertices among all graphs Γ such that Aut $\Gamma \cong G$.

Value of $\alpha(G)$

Theorem

The value of $\alpha(G)$ has been established for the following groups G:

- *finite abelian groups (Arlinghaus);*
- *hyperoctahedral groups (Haggard, McCarthy, Wohlgemuth);*
- *symmetric groups (Quintas);*
- *alternating groups of degree at least 13 (Liebeck);*
- *generalized quaternion groups (Graves, Graves, Lauderdale); and*
- *dihedral groups (Graves, Graves, Haggard, Lauderdale, McCarthy).*

Vertex-Minimal Planar Graphs

Marušič was the first to consider vertex-minimal *planar* graphs with a prescribed automorphism group.

Definition

For a finite group G, we let $\boldsymbol{\alpha^P(G)}$ denote the minimum number of vertices among all planar graphs Γ such that $\operatorname{Aut}\Gamma \cong G$. If no planar graph Γ satisfies $\operatorname{Aut}\Gamma \cong G$, then we define $\alpha^P(G) = \infty$.

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It is clear that $\alpha(G) \leq \alpha^P(G)$ for all finite groups G.

Let $\mathbb{Z}_m$ denote the cyclic group of order m.

Theorem (Marušič, 1980)

Assume $m = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$ is the prime factorization of the integer m, where $a_i \geq 1$ for all $i \in [k]$. If m is odd, then

$$\alpha^P(\mathbb{Z}_m) = 3(p_1^{a_1} + p_2^{a_2} + \ldots + p_k^{a_k}).$$

Vertex-Minimal Planar Graphs

Marušič conjectured that a similar result holds when $m = 2^n$. Archer et al. proved Marušič's conjecture with the following theorem.

Theorem (Archer, Darby, Lauderdale, Linson, Maxfield, Schmidt, Tran, 2017)

If $m = 2^s$ with $s \geq 1$, then

$$\alpha^P(\mathbb{Z}_m) = \begin{cases} 2 & \text{if } s = 1 \\ 2m+2 & \text{if } s > 1. \end{cases}$$

Vertex-Minimal Planar Graphs

Marušič conjectured a similar result for when m is an even and not a power of 2. We proved Marušič's conjecture, which is stated in the following theorem.

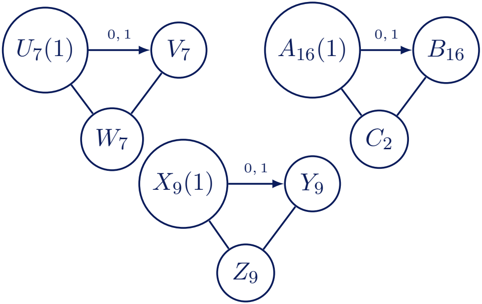
Theorem (Jones, L., T., 2018)

If $m = 2^s \cdot p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$ is the prime factorization of the integer m, where $a_i \geq 1$ for all $i \in [k]$ and $k \geq 1$, then

$$\alpha^P(\mathbb{Z}_m) = \begin{cases} 2 + 3(p_1^{a_1} + p_2^{a_2} + \ldots + p_k^{a_k}) & \text{if } s = 1 \\ 2 + 2^{s+1} + 3(p_1^{a_1} + p_2^{a_2} + \ldots + p_k^{a_k}) & \text{if } s > 1. \end{cases}$$

Example of Theorem:

Our results prove that $\alpha^P(\mathbb{Z}_{1008}) = 82$.
Since $1008 = 2^4 \cdot 3^2 \cdot 7$, then $\Gamma_{1,008} = \Gamma'_{16} + \Gamma'_9 + \Gamma'_7$.



F^*-diagram of $\Gamma_{1,008}$

Connected Cyclic Graphs

Marušič conjectured that the connected cyclic vertex-minimal graph would have just one additional vertex.

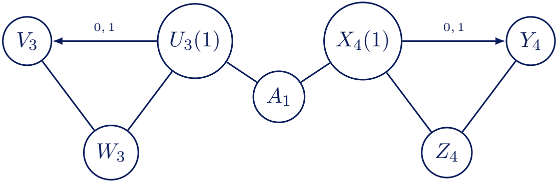
Theorem (Jones, L., T., 2018)

If $m = 2^s \cdot p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$ *is the prime factorization of the integer* m*, where* $a_i \geq 1$ *for all* $i \in [k]$ *and* $k \geq 1$*, then*

$$\alpha_c^P(\mathbb{Z}_m) = \begin{cases} 3(p_1^{a_1} + p_2^{a_2} + \ldots + p_k^{a_k}) + 3 & \text{if } s = 1 \\ 3(2^s + p_1^{a_1} + p_2^{a_2} + \ldots + p_k^{a_k}) + 1 & \text{if } s > 1. \end{cases}$$

Example $\alpha_c^P(\mathbb{Z}_{12})$

Our results prove that $\alpha_c^P(\mathbb{Z}_{12}) = 22$.



Dihedral Groups

Definition

A **dihedral group** has presentation

$$D_{2n} = \langle r, s : r^n = 1 = s^2, rs = sr^{-1} \rangle,$$

and is often thought of as the symmetries of a regular n-gon.

The value of $\alpha^P(D_{2n})$ denotes the order of a vertex-minimal planar graph with dihedral symmetry.

Dihedral Groups

We know the lower bound of $\alpha^P(D_{2n})$ is $\alpha(D_{2n})$, and the values of $\alpha(D_{2n})$ were found by Haggard, McCarthy, Graves, Graves and Lauderdale.

Additionally we know the upper bound corresponds to the order of an n-gon. An n-gon is planar.

Dihedral Groups

Let's look at an example where $\alpha(D_{2n}) \neq \alpha^P(D_{2n})$.
Assume $n = 136, 191, 250$.

Dihedral Groups

Let's look at an example where $\alpha(D_{2n}) \neq \alpha^P(D_{2n})$. Assume $n = 136,191,250$. In this case,

$$\alpha(D_{2n}) = 1,044,$$

and

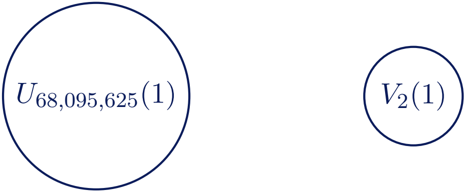
$$\alpha^P(D_{2n}) = 68,095,625 + 2.$$

Dihedral Groups

Lemma

Let D_{2n} be a dihedral group, where n is twice an odd number. Then $D_{2n} \cong D_n \times \mathbb{Z}_2$. Otherwise D_{2n} is directly indecomposable.

For $n = 136,191,250$ we now have two components, representing $D_{68,095,625}$ and $\mathbb{Z}_2$.



Dihedral Groups

Theorem (Jones, L., T., 2018)

Let D_{2n} be a dihedral group for some $n \geq 3$. If n is twice an odd integer, then $\alpha^P(D_{2n}) = \frac{n}{2} + 2$. Otherwise, $\alpha^P(D_{2n}) = n$.

Notice $n = 136, 191, 250$ falls into the special case.

Dihedral Groups

Theorem (Jones, L., T., 2018)

Let D_{2n} be a dihedral group for some $n \geq 3$. If n is twice an odd integer, then $\alpha^P(D_{2n}) = \frac{n}{2} + 2$. Otherwise, $\alpha^P(D_{2n}) = n$.

Notice $n = 136, 191, 250$ falls into the special case.Thus proving the order of a vertex-minimal planar graph with dihedral symmetry.

Future Work

Theorem (Jones, L., T., 2018)

Let p be a prime number.
1. *If $p = 2$, then $\alpha^P(\mathbb{Z}_p \times \mathbb{Z}_p) = 4$.*
2. *If $p > 2$, then $\alpha^P(\mathbb{Z}_p \times \mathbb{Z}_p) = 6p$.*

Open Question.

What is the value of $\alpha^P(G)$, when G is a finite noncyclic abelian group?

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Future Work

Conjecture (Jones, L., T., 2018).

Assume that $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times ... \times \mathbb{Z}_p$, where n is the number of times $\mathbb{Z}_p$ appears and p is an odd prime. If n is even, then

$$\alpha^P(G) = 2\sum_{i=1}^{\frac{n}{2}}(i+2)p$$

and if n is odd, then

$$\alpha^P(G) = \left(\frac{n+5}{2}\right)p + 2\sum_{i=1}^{\frac{n-1}{2}}(i+2)p.$$

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Gratitude

Thank You!

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