Pattern Avoidance in Acyclic Digraphs

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Nebraska Conference for Undergraduate Women in Mathematics

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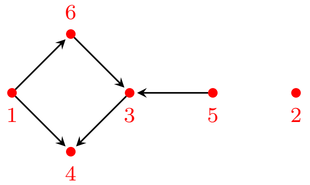
Directed Graph (Digraph):

- **Vertices**
- (Directed) Edges
  - Tail $\rightarrow$ Head



Definitions: Graph Theory

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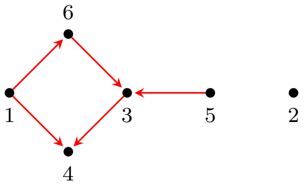

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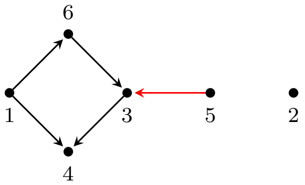
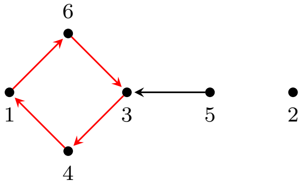


- An *acyclic digraph* is a directed graph that has no (directed) cycles.



Cyclic Digraph

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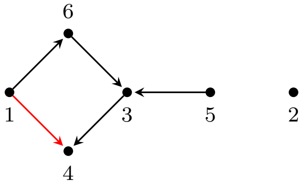

Acyclic Digraph

Definitions: Graph Theory

- **A *source* is a vertex with no incoming edges.**
- A *sink* is a vertex with no outgoing edges.
- A *descent* is a directed edge $x \to y$ such that $x > y$.
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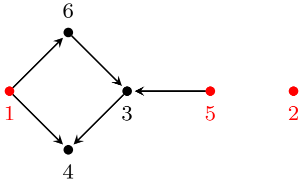

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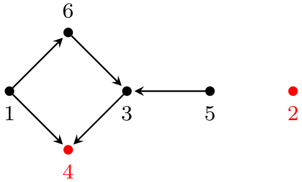

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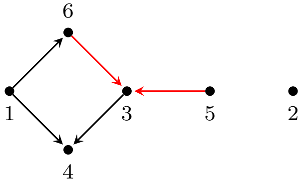

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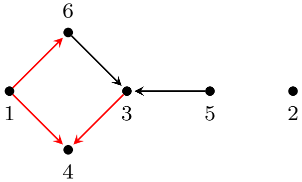

Pattern Avoidance

Patterns (length 3): 123, 132, 213, 231, 312, 321

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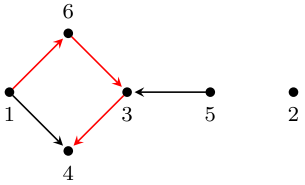
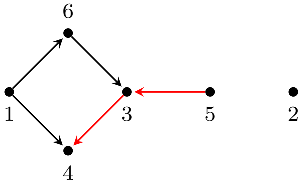
Paths in Acyclic Digraphs





- 1, 6, 3, 4
- 5, 3, 4

Patterns:

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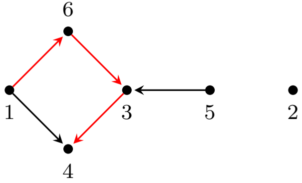
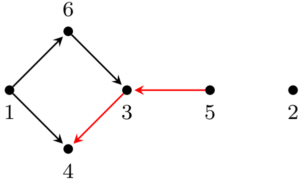
Paths in Acyclic Digraphs





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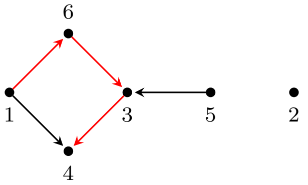

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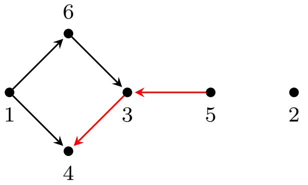

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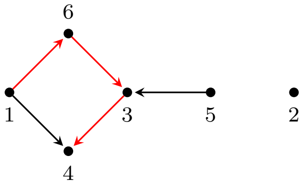

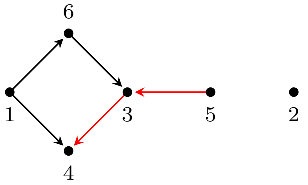

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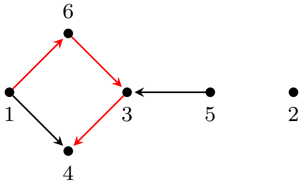

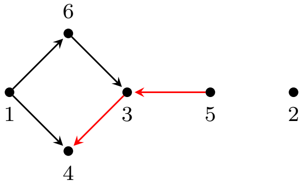

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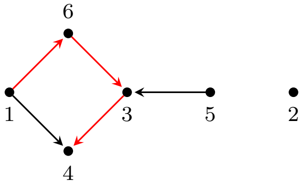

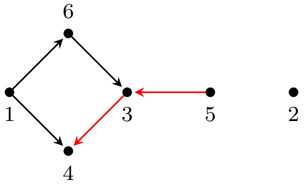

- 1, 6, 3, 4

Patterns: 123, 132, 312

- 5, 3, 4

Patterns avoided: 213, 231, 321

What is known about acyclic digraphs and pattern avoidance?

- Number of *labeled* acyclic digraphs (Robinson 1973, Stanley 1973)
- Number of *unlabeled* acyclic digraphs (Robinson 1977)
- Enumerated based on number of descents (Archer, Graves 2017)
  - Acyclic digraphs avoiding 12 or 21: $2^{\binom{n}{2}}$
- Number of permutations which avoid certain sets of patterns (Simon, Schmidt 1985)

Example: Avoiding {213, 231, 312, 321}

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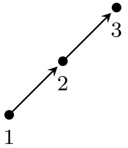
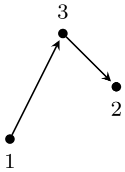
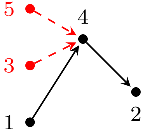




- Head of descent must be greater than vertices which go to tail of descent
  - $1 \to 4 \to 2$: 132
  - $3 \to 4 \to 2$: 231
  - $5 \to 4 \to 2$: 321



Example: Avoiding {213, 231, 312, 321}

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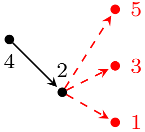
- Only patterns allowed: 123, 132

- Head of descent must be a sink
  - $4 \to 2 \to 1$: 321
  - $4 \to 2 \to 3$: 312
  - $4 \to 2 \to 5$: 213



Example: Avoiding {213, 231, 312, 321}

Theorem (Liang, Martinez, Osterman, 2018)

$$\mathrm{Av}_n(213, 231, 312, 321) = \sum_{s=1}^{n} p_{n,s}$$

where $p_{n,s} = \sum_{c=0}^{n-s} p_{n-1,s-1+c} \binom{s-1+c}{c} 2^{n-s-c} + p_{n-1,s}(2^s - 1)$

with initial condition $p_{1,1} = 1$.

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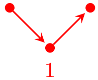
- $p_{n,s}$: $\mathrm{Av}_n(213, 231, 312, 321)$ with s sources
- Recursion by adding 1 to graph with $n-1$ vertices
    - Source
    - Sink



Example: 1 is a source ($\mathrm{Av}_n(213, 231, 312, 321)$)

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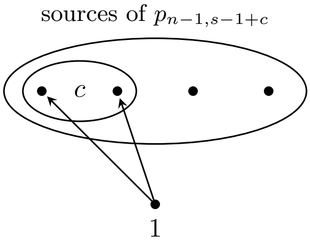

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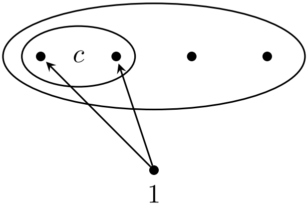
- Any edge from 1 to another vertex: never creates pattern we avoid
  - Choose c sources to add edges from 1
  - Add edges from 1 to any non-source vertex

sources of $p_{n-1,s-1+c}$



Example: 1 is a sink ($\text{Av}_n(213, 231, 312, 321)$)

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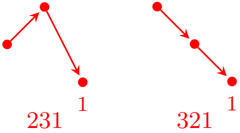

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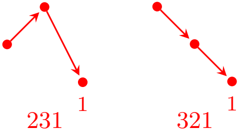
- Only edges from sources allowed
- At least 1 edge to prevent double-counting from source case



Comparisons between Avoidance Classes

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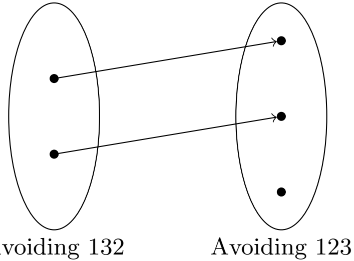
- Begin with graph in one avoidance class (i.e. avoiding 132)



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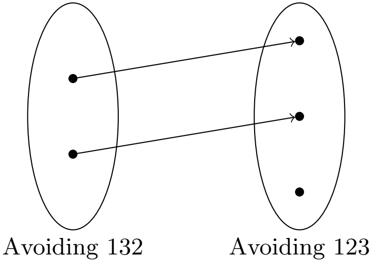
- Begin with graph in one avoidance class (i.e. avoiding 132)
- Use algorithm to change graph to one of another avoidance class (i.e. avoiding 123)



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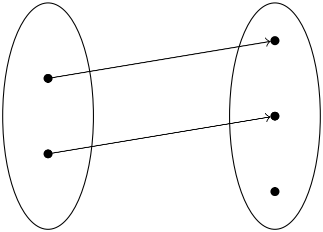
- Prove process gives a different new graph for each old graph (one-to-one)



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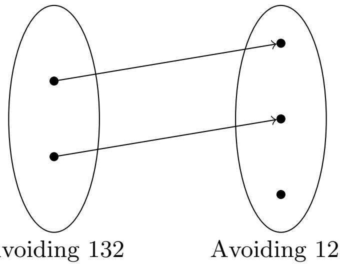

- Number of graphs in first group $\leq$ number of graphs in second group

Comparisons between Avoidance Classes

Theorem (Hopkins, Weiler 2015)

The number of permutations of any partially ordered set that avoid the pattern 123 *is greater than or equal to the number of those that avoid* 132.

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Corollary (Liang, Martinez, Osterman, 2018)

$\mathrm{Av}_n(132) \leq \mathrm{Av}_n(123)$

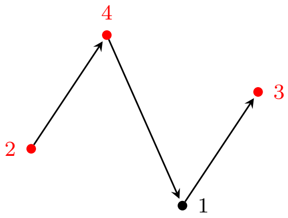
Summary of Results

Enumerations for Non-Consecutive Patterns	
Avoidance Set	**Enumerated By**
{123} {132}	comparison (no general formula)
{132, 231}	closed form
{123, 213} {132, 213}	comparison (no general formula)
{123, 213, 312}	closed form
{123, 132, 231, 321}	recurrence relation
{123, 132, 213, 231}	recurrence relation
{123, 132, 231, 312}	comparison (above and below, no general formula)
{132, 213, 231, 312}	recurrence relation
{132, 213, 231, 312, 321}	recurrence relation
{123, 213, 231, 312, 321}	recurrence relation

Consecutive Pattern Avoidance

Definition

A *consecutive* pattern in an acyclic digraph is a pattern such that all pairs of consecutive vertices in the pattern are adjacent in the graph.



No consecutive 132 pattern.

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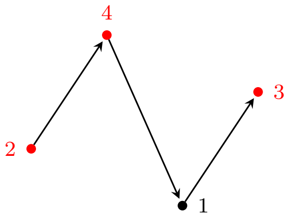
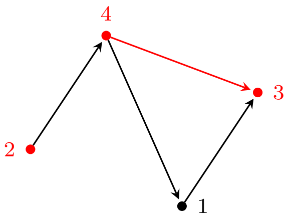


No consecutive 132 pattern.



$2 \to 4 \to 3$ is an instance of a consecutive 132 pattern.

Consecutive Pattern Avoidance

Enumerations for Consecutive Patterns	
Avoidance Set	**Enumerated By**
$\{132, 231\}$	closed form
$\{123, 132\}$	recurrence relation
$\{123, 132, 321\}$	recurrence relation
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Open Questions and Future Work

- Enumerations for remainder of length 3 pattern avoidance
- Comparisons between more pairs of avoidance classes, which may include additional methods for comparisons
- Enumerations by number of peaks, i.e., occurrences of an ascent followed by a descent

Acknowledgments

Many thanks to Dr. Christy Graves for her guidance on this project, as well as to coworkers Xuming Liang and Vaughn Osterman. This project was funded by NSF grant 1659221 for the REU at the University of Texas at Tyler.

THANK YOU!