Chromatic Symmetric Functions with respect to Complete Graphs

Sofía Martínez Alberga
in collaboration with J. Kazdan, L. Kröll,
O. Melnyk, and A. Tenenbaum

University of California, Riverside

smart040@ucr.edu

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Outline

1. Background
2. Motivation
3. E-positivity

Graph Theory: Definitions

- Let $\Gamma = (V, E)$ be a **graph** on n vertices
- Formally we can define a **coloring**, of Γ, via a map $\kappa : V \to \mathbb{N}$
- **Proper colorings of** Γ are the colorings of vertices for which any two adjacent vertices have different colors.

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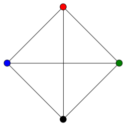

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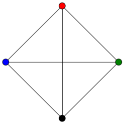

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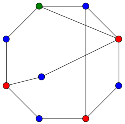

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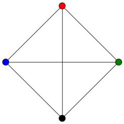

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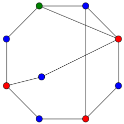

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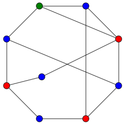

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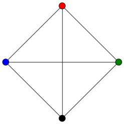

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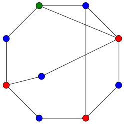

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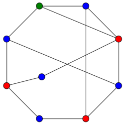

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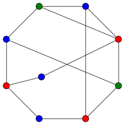

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- A graph, Γ, is said to be free of a subgraph, i.e **Δ- free** , if Δ is *not* a induced subgraph of Γ.

Subgraph: Example

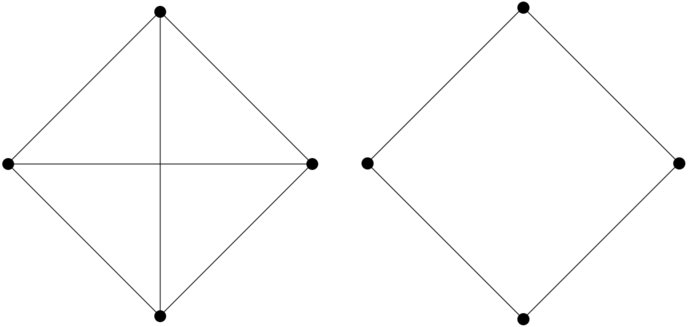

Figure 3: K_4 and C_4

If the original graph is K_4, then C_4 is only a subgraph of K_4 *not* an induced subgraph.

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Algebra: the Ring of Symmetric Functions

The **ring of symmetric functions** is denoted as Λ and is defined as

$$\Lambda = \mathbb{C}m_\lambda$$

from which we get that the space Λ^n has basis

$$\{m_\lambda : \lambda \vdash n\}$$

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- Why E-Positivity?
  - E-Positivity is an invariant which allows for classifying to take place.

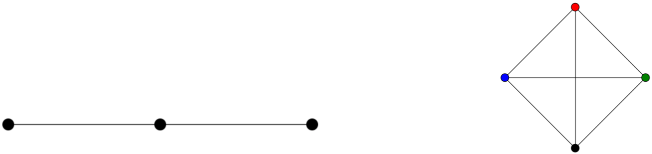
Chromatic Symmetric Functions

- **Chromatic Symmetric Function, (CSF)**, of Γ :

$$X_\Gamma = \sum x_{\kappa(v_1)} x_{\kappa(v_2)} \ldots x_{\kappa(v_n)}$$

where the sum is over all proper colorings of Γ with colors from the positive integers and $v_i \in V$

A Few Examples



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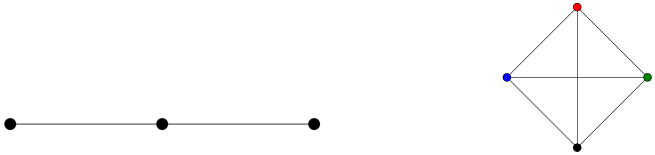

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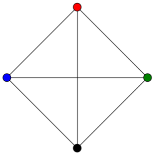

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- *Example:* K_n only has colorings where all vertices are of different colors. Then we get:

$$X_{K_n} = n!e_n$$

Outline

Some Major Questions in Algebraic Combinatorics

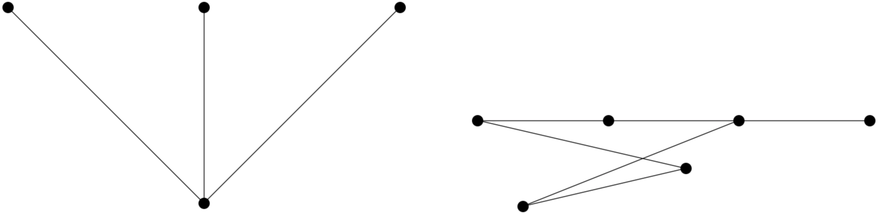

Figure 5: Claw and Incomparibility Graph

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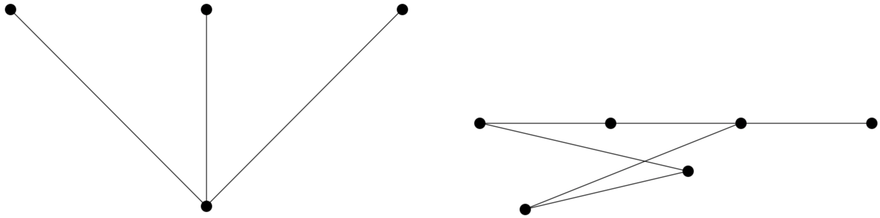

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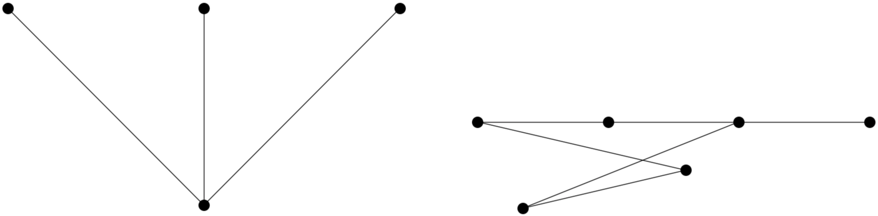

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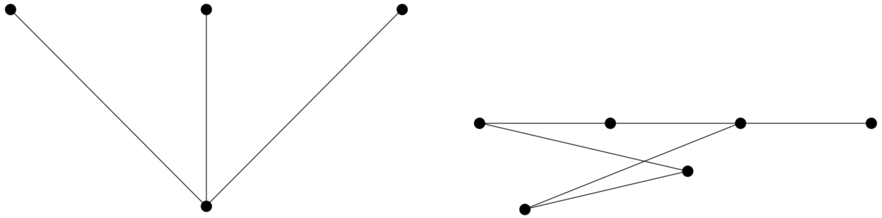

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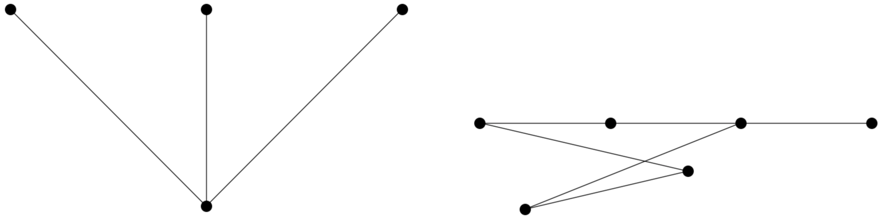

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- What families of graphs are e-positive?
  $\rightarrow$ Claw-free, incomparability graphs
- Can graphs be distinguished by their chromatic symmetric functions?
  $\rightarrow$ Trees : a graph with no cycles

Motivation

- Classifying graphs by their CSF.
- Which graph classes are e-positive?
- Identifying properties encoded in the CSF.
- Which families of graphs are uniquely determined by there CSF.

Our Research

Ways We Proved:
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- Explicit Formula
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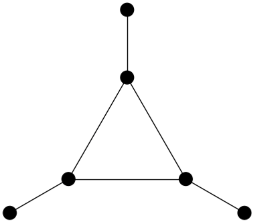
What we proved was uniquely determined but CSF:

- Generalized Spiders

Outline

What is the net and Why the net?



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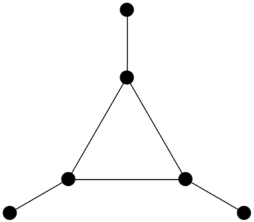
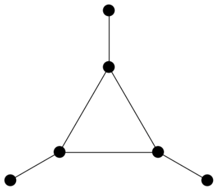


Figure 6: Net

$$X_N = 6e_{3,2,1} - 6e_{3,3} + 6e_{4,1,1} + 12e_{4,2} + 18e_{5,1} + 12e_6$$

Stanley found a claw-free non e-positive graph known as the net.

Nets and Generalized Nets



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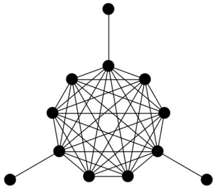

(b) Generelized Net

Figure 7

Formula for CSF of generalized nets

Our approach:

- Count all proper colorings
- Get CSF in terms of monomial symmetric functions
- Express monomials in terms of elementary symmetric functions
- Get CSF in terms of elementary symmetric functions

Count Proper Colorings

From the structure of the graph we get that all proper colorings are:

- $(1, 1, ..., 1)$
- $(2, 1, 1, ..., 1)$
- $(2, 2, 1, ..., 1)$
- $(2, 2, 2, 1, ..., 1)$
- $(3, 1, ..., 1)$
- $(3, 2, 1, ..., 1)$
- $(4, 1, 1, ..., 1)$

Proper Colorings Pictorally

Example Case

$(3, 1, ..., 1)$ has $(3n - 5)n!$ colorings

Figure 8: Possible arrangements of (3,1,...,1)

First arrangement gives us $n!$ choices
The second one gives us $3(n - 2)n!$

CSF in m-basis

Every coloring gives us a term in monomial basis. When put it together we get:

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$$
\begin{aligned}
X_G = (n+3)!m_{(1,1,1,\ldots,1)} + 3n(n+1)!m_{(2,1,1,\ldots,1)} + \\
+ 6(n^2-2n+2)(n-1)!m_{(2,2,1,\ldots,1)} + \\
+ 6(n^3-6n^2+14n-13)(n-3)!m_{(2,2,2,1,\ldots,1)} + \\
+ (3n-5)n!m_{(3,1,1,\ldots,1)} + 3(n^2-4n+5)(n-2)!m_{(3,2,1,\ldots,1)} + \\
+ (n-3)(n-1)!m_{(4,1,1,\ldots,1)}
\end{aligned}
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CSF in e-basis

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\begin{aligned}
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& + 6(n-1)(n-3)! e_{(n+1,2)} + 3(n^2-2n-1)(n-2)! e_{(n+1,1,1)} + \\
& + 6(n-2)! e_{(n,2,1)} - 6(n-3)! e_{(n,3)} + (n-3)(n-1)! e_{(n,1,1,1)}
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$$X_G = (n+3)(n-1)!e_{(n+3)} + 3(n^2-3)(n-2)!e_{(n+2,1)} + \\ + 6(n-1)(n-3)!e_{(n+1,2)} + 3(n^2-2n-1)(n-2)!e_{(n+1,1,1)} + \\ + 6(n-2)!e_{(n,2,1)} - 6(n-3)!e_{(n,3)} + (n-3)(n-1)!e_{(n,1,1,1)}$$

Conclusion

Generalized nets are not e-positive

References I

S. Cho and J. Huh
On e-Positivity and e-Unimodality of Chromatic Quasisymmetric Functions.
arXiv:1711.07152, 2017.

V. Gasharov
Incomparability graphs of (3+1)-free posets are s-positive.
Discrete Mathematics 157, 193-197, 1996.

C.D. Godsil
Algebraic Combinatorics.
Chapman and Hall Inc., New York, 1993.

I. Gutman and M. Lepoviĉ
No starlike trees are cospectral.
Discrete Mathematics 242, 291-295, 2002.

B. Sagan.
The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions.
Springer, 2001.

References II

R. Stanley.
A Symmetric Function Generalization of the Chromatic Polynomial of a Graph
Advances in Mathematics, 111, 166-194, (1995).

Thank You

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Questions