The Shape of Large Soap Bubbles
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• Belgian Physicist, 1801-1883
• Studied capillary action, surface tension, and minimal surfaces
• Observed soap bubbles and films to form Plateau’s Laws
• Plateau’s Problem: Find a surface containing a specified volume that minimizes surface area.
Plateau’s Problem

Small soap bubbles are a physical manifestation of the solution.


**Large Soap Bubbles**

- Increased material needed to contain large volume gives significant mass for gravity to act upon, flattening large bubbles

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Expected spherical form
Experimentally obtained bubble
Applications

Large inflatable structures, like Festo’s Airquarium!
Formulating the Model

Principle of Virtual Work – Static systems minimize potential energy

Bubbles take the shape that minimizes potential energy

Find the equation for the shape of bubbles of a given volume with the least potential energy!
How do we do this?

Functionals: Map a function to a real number
- Create a functional that represents the total potential energy of the bubble to be minimized

Calculus of variations: Find the extrema of functionals
- Minimize our functional and produce equations governing the shape of the surface
Parameterization

\[(u, v, z(u, v))\]
Constructing the functional

• Interfacial Free Energy: Difference between cohesive and adhesive forces at the membrane.

\[ \gamma \iiint_{\Omega} |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv \]

• Gravitational Energy: Effect of gravity on the membrane.

\[ \beta \iiint_{\Omega} z(u, v) |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv \]
We can remove the effect of interfacial free energy with a vertical translation.

\[ E[z(u, v)] = \gamma \iint_{\Omega} |r_u \times r_v| \, dudv + \beta \iint_{\Omega} z(u, v) |r_u \times r_v| \, dudv. \]

Combine Terms

\[ E[z(u, v)] = \beta \iint_{\Omega} \left( \frac{\gamma}{\beta} + z(u, v) \right) |r_u \times r_v| \, dudv. \]

Make substitution such that \( z(u, v) = \zeta(u, v) - \frac{\gamma}{\beta}. \)

\[ E[\zeta(u, v)] = \beta \iint_{\Omega} \zeta(u, v) |r_u \times r_v| \, dudv. \]
Why is this interesting?

This means studying soap bubbles is like studying heavy surfaces!
Translationally Invariant Surfaces

To simplify analysis, we reparametrize our surface such that $\mathbf{r} = (u, v, \zeta(u))$. 
New Equation with Volume Constraint

\[ E[\zeta(u)] = \int_{-a}^{a} \zeta(u) \sqrt{1 + (\zeta(u)')^2} du + \Lambda \left( \int_{-a}^{a} \zeta(u) du - A_0 \right) \]
The Model

With our chosen parameterization, we obtain the following governing equation from the Euler-Lagrange equation.

\[ \frac{\zeta''}{(1 + (\zeta')^2)^{3/2}} = \frac{1}{\sqrt{1 + (\zeta')^2}} + \Lambda. \]
Result 2

There are four distinct one-parameter families of curves for translationally invariant bubbles.
Where are the bubbles?

Pick two endpoints. The curve between is the shape of the bubble!
Where are the bubbles?

Not all choices of endpoints produce physical solutions.
## Future Work

<table>
<thead>
<tr>
<th>Classify</th>
<th>Classify critical points to confirm which solutions are minimizing equations</th>
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</thead>
<tbody>
<tr>
<td>Analyze</td>
<td>Perform stability analysis to determine which forms bubbles will take</td>
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<tr>
<td>Extend</td>
<td>Extend this analysis to rotationally symmetric bubbles</td>
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Acknowledgements

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Further Reading


Thank you for listening!

Any questions?

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