Ineffective Sets and the Region Crossing Change

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Knot Theory Basics

A knot is a proper embedding of a closed curve in $\mathbb{R}^3$.

A link of $m$ components is a proper embedding of $m$ closed curves in $\mathbb{R}^3$.

A link diagram is a regular projection together with crossing information.

There are 2 $c$-diagrams associated to a projection with $c$ crossings.
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![Projection](image1.png) ![Diagram](image2.png)
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There are $2^c$ diagrams associated to a projection with $c$ crossings.
Region Crossing Change (RCC) - RCC is an operation on a link diagram in which a region is selected and all crossings incident to that region are reversed.

\[ \text{RCC} \rightarrow \]

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Previous Results

RCC is an unknotting operation. All knot diagrams having the same underlying projection are RCC-equivalent.

(Cheng-Gao) Provide necessary and sufficient conditions for a link diagram to be RCC-equivalent to an unlink diagram.

(Dasbach-Russell) Count RCC-equivalence classes for link projections on closed, orientable surfaces such as the torus.
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Motivating Question

Given a pair of RCC-equivalent diagrams, what is the minimum number of RCCs needed to transform one diagram into the other?

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An Example

Note that every crossing must be changed. Therefore, we need to select an odd number of regions around every crossing. There are four ways to do this.

Conclusion: The RCC-distance between the diagrams is two.
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Ineffective Sets

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- Let $R$ be an ineffective set and $S$ be an arbitrary set of regions. Then, $R \oplus S$ has the same effect as $S$. 

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**Lemma (Cheng & Gao)**

An $m$-component link diagram has $2m + 1$ ineffective sets.
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\text{(Diagram: Ineffective Sets)}
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**Lemma (Cheng & Gao)**

An $m$-component link diagram has $2^{m+1}$ ineffective sets.
A reducible crossing of a link is bordered on two sides by the same region. A reducible diagram has at least one reducible crossing. Reducible crossings complicate the study of RCC equivalence.
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**RCC and Reducible Crossings**

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- A **reducible diagram** has at least one reducible crossing.
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Reducible crossings complicate the study of RCC equivalence.
A **checkerboard coloring** is a black ($B$) and white ($W$) coloring of a projection such that opposite regions are the same color and adjacent regions are opposite colors.
Ineffective Sets of Reduced Links

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Given a checkerboard shading of a reducible link projection, at most one of $B$ or $W$ is ineffective.
Reducible Diagrams

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Tricoloring

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- every reduced crossing is checkerboard shaded by two of the three colors,
- every reducible crossing is bordered by three regions of different colors.
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Ineffective Sets for Arbitrary Knot Projections

**Theorem**

Given a tricoloring of a link projection, the sets $\emptyset, B \sqcup W, B \sqcup G,$ and $W \sqcup G$ are ineffective.
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For a knot projection, these are the only ineffective sets.
Each $m$-component link has $2^{m+1}$ ineffective sets.

Basis:

\[
\{ \ldots \}
\]

Ineffective sets:

\[
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Link Projections

- Each $m$-component link has $2^{m+1}$ ineffective sets.
- A basis for these sets consists of:

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Thank you for your attention and the NCUWM staff for organizing this conference!