Ineffective Sets and the Region Crossing Change

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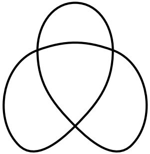
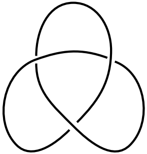
Projection: 

Diagram: 

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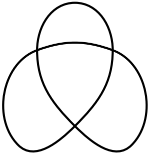
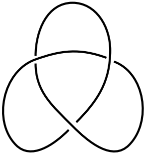
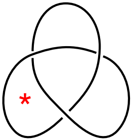
Projection: 

Diagram: 

There are 2^c diagrams associated to a projection with c crossings.

Region Crossing Change

Region Crossing Change (RCC) - RCC is an operation on a link diagram in which a region is selected and all crossings incident to that region are reversed.



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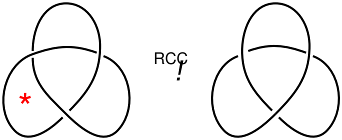

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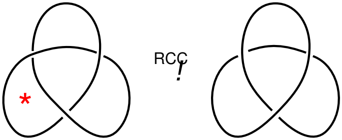

Two diagrams are **RCC-equivalent** if one can be obtained from the other via a sequence of RCCs.

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- (Cheng-Gao) Provide necessary and sufficient conditions for a link diagram to be RCC-equivalent to an unlink diagram.
- (Dasbach-Russell) Count RCC-equivalence classes for link projections on closed, orientable surfaces such as the torus.

Motivating Question

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We call this the RCC-distance between diagrams.

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Conclusion : The RCC-distance between the diagrams is two.

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Lemma (Cheng & Gao)

An m-component link diagram has 2^{m+1} ineffective sets.

RCC and Reducible Crossings

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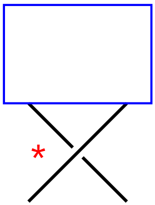

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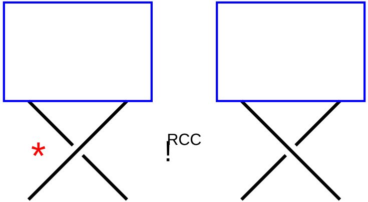

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For a knot projection, these are the only ineffective sets.

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Thank you for your attention and the NCUWM staff for organizing this conference!