Spectral Characterizations of Anti-Regular Graphs

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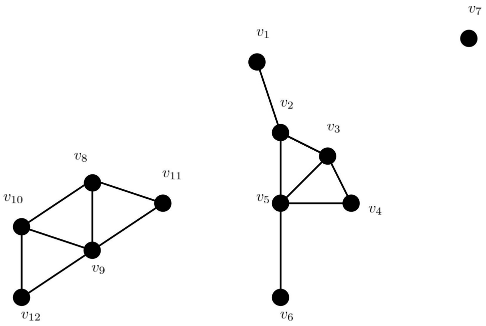
Overview

Basic Graph Theory



- $G = (V, E)$ with $|V| = n = 12$ and $|E| = m = 14$
- Vertex 9 is adjacent to vertex 11; $9 \sim 11$
- Degree of vertex 9, $\deg(9) = 4$
- Isolated vertex, no dominating vertex
- Degree sequence of the graph: $d(G) = (4, 4, 3, 3, 3, 3, 2, 2, 2, 1, 1, 0)$
- Disconnected graph

The Adjacency Matrix

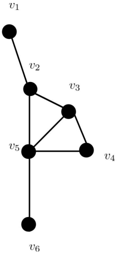
Definition

Given a graph $G = (V, E)$, the adjacency matrix A is the matrix whose (i, j) entry is 1 if $v_i \sim v_j$ and 0 otherwise.

- Adjacency is commutative, so $v_i \sim v_j$ also means $v_j \sim v_i$
- A is then symmetric

The Adjacency Matrix

- Consider the graph $G = (V, E)$



- The adjacency matrix A of G is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The Anti-Regular Graph

Definition

A graph G is said to be k-**regular** if every vertex has degree k.

Definition

The **anti-regular graph** G is the graph such that only two vertices have the same degree.

- The anti-regular graph is a type of threshold graph

Definition

A **threshold graph** is a graph created by repeatedly adding either isolated or dominating vertices to a graph.

- -1 and 0, or λ_0, are trivial eigenvalues of the anti-regular graph
- The interval $[-1, 0]$ was known to be a forbidden interval for the eigenvalues of all threshold graphs

The Anti-Regular Graph

Figure: The anti-regular graph on 1 vertex

Figure: The anti-regular graph on 2 vertices

The Anti-Regualr Graph

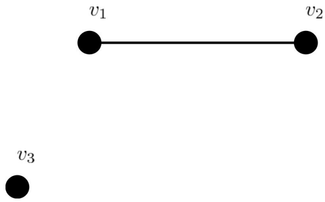


Figure: The anti-regular graph on 3 vertices

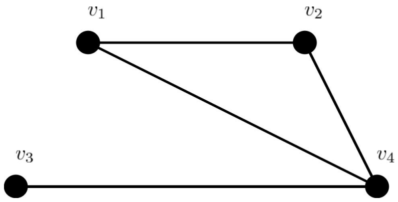


Figure: The anti-regular graph on 4 vertices

The Anti-Regualr Graph

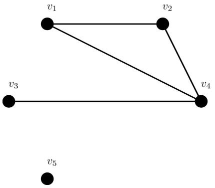


Figure: The anti-regular graph on 5 vertices

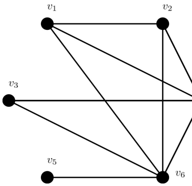


Figure: The anti-regular graph on 6 vertices

The Anti-Regular Graph

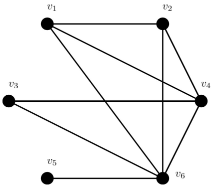


Figure: The anti-regular graph on 6 vertices

This graph has adjacency matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Analysis of our Findings

Theorem

Let $n = 2k$ and let A_n denote the connected anti-regular graph with n vertices. Then λ is an eigenvalue of A_n if and only if

$$\lambda = \frac{\sin k\theta}{\sin k\theta + \sin{(k-1)\theta}}$$

where $\theta = \arccos\left(\frac{1 \quad 2\lambda \quad 2\lambda^2}{2\lambda(\lambda+1)}\right)$.

Analysis of our Findings

- Recall

$$\theta = \arccos\left(\frac{1-2\lambda-2\lambda^2}{2\lambda(\lambda+1)}\right).$$

By the fact that the domain of arccos is $[-1, 1]$,

Theorem

Let A_n be the adjacency matrix of the connected anti-regular graph with n vertices. The only eigenvalue of A_n in the interval $\Omega = \left[\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}\right]$ is $\lambda_0 \in \{0, -1\}$.

Analysis of our Findings

- Recall $\lambda = \frac{\sin k\theta}{\sin k\theta + \sin{(k \quad 1)\theta}}$ and $\theta = \arccos\left(\frac{1 \quad 2\lambda \quad 2\lambda^2}{2\lambda(\lambda+1)}\right)$.

Eigenvalue Equation 1

$$F(\theta) = \frac{\sin k\theta}{\sin k\theta + \sin{(k-1)\theta}}$$

Eigenvalue Equation 2

$$f_1(\theta) = \frac{-(\cos\theta + 1) + \sqrt{(\cos\theta + 1)(\cos\theta + 3)}}{2(\cos\theta + 1)}$$

$$f_2(\theta) = \frac{-(\cos\theta + 1) - \sqrt{(\cos\theta + 1)(\cos\theta + 3)}}{2(\cos\theta + 1)}.$$

Analysis of our Findings

Eigenvalues

- Positive eigenvalues obtained by $f_1(\theta) = F(\theta)$
- Negative eigenvalues obtained by $f_2(\theta) = F(\theta)$

Singularities

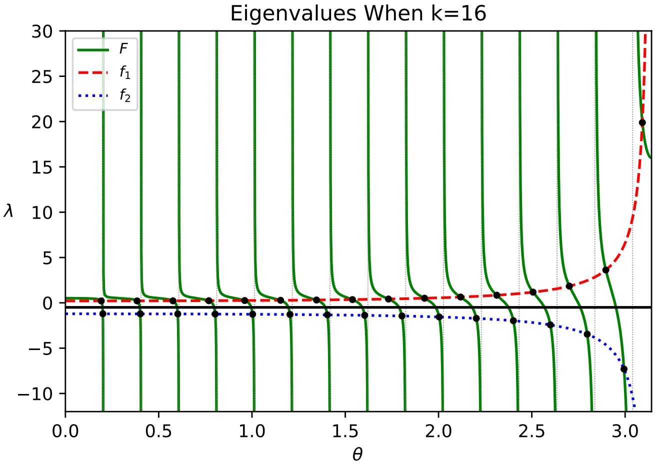
Since

$$F(\theta) = \frac{\sin k\theta}{\sin k\theta + \sin{(k-1)\theta}},$$

singularities occur where

$$\sin k\theta + \sin(k-1)\theta = 0.$$

Analysis of our Findings

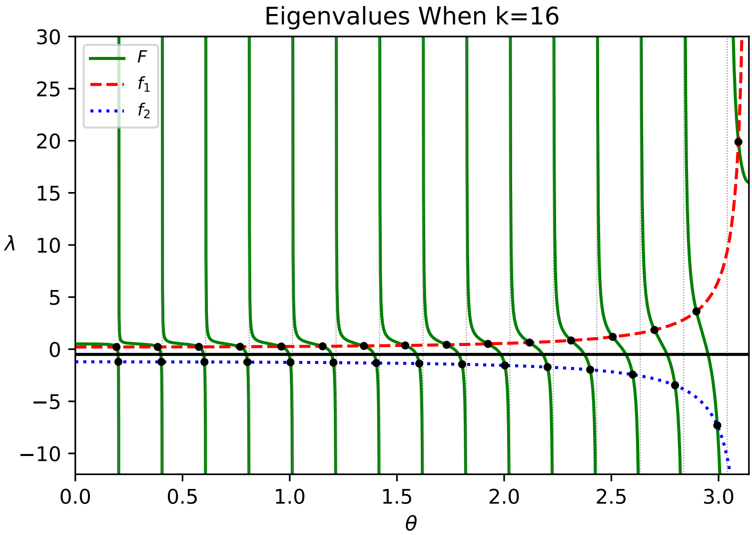


- Singularities occur at $\theta = \frac{2j\pi}{2k\ 1}$ for $j \in \{1, 2, \ldots, k-1\}$.

Analysis of our Findings

- Singularities at $\theta = \frac{2j\pi}{2k \quad 1}$
- Eigenvalues are approximately symmetric about $\frac{1}{2}$

Analysis of our Findings



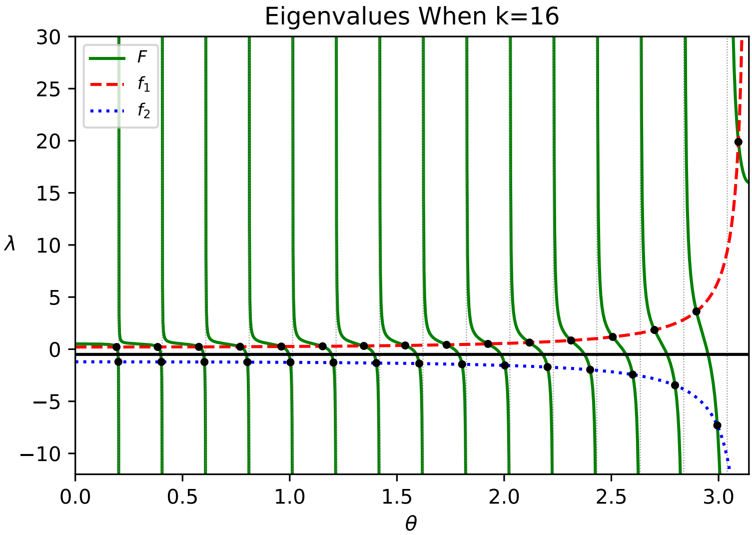
Analysis of our Findings

- Singularities at $\theta = \frac{2j\pi}{2k-1}$

- Eigenvalues are approximately symmetric about $-\frac{1}{2}$

- As $k \to \infty$, the eigenvalues approach the Ω bound, $\left[\frac{-1-\sqrt{2}}{2}, \frac{-1+\sqrt{2}}{2}\right]$

Analysis of our Findings



Analysis of our Findings

Theorem

Let $\lambda_1^+(k)$ denote the smallest positive eigenvalue of A_n and let $\lambda_1^-(k)$ denote the negative eigenvalue of A_n closest to the trivial eigenvalue λ_0. The following hold:

i) *The sequence $\{\lambda_1^+(k)\}_{k=1}^{\infty}$ is strictly decreasing and converges to $\frac{1+\sqrt{2}}{2}$.*

ii) *The sequence $\{\lambda_1^-(k)\}_{k=1}^{\infty}$ is strictly increasing and converges to $\frac{1-\sqrt{2}}{2}$.*

- Note that therefore, the interval bound $\Omega = \left[\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}\right]$ is a sharp bound.

Conjectures

Conjecture

For any n, the anti-regular graph A_n has the smallest positive eigenvalue and has the largest non-trivial negative eigenvalue among all threshold graphs on n vertices.

Conjecture

Other than the trivial eigenvalues $\{0, -1\}$, the interval $\Omega = \left[\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}\right]$ does not contain an eigenvalue of any threshold graph.

- These conjectures were recently proven in "Eigenvalue-free interval for threshold graphs, arXiv.:1807.10302, 2018" by Ebrahim Ghorbani.

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Thank you for listening!

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