Intersections of Shortest Taxicab Paths in the Sierpiński Carpet

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Evangel University, Lafayette College, CU Boulder, Haverford College

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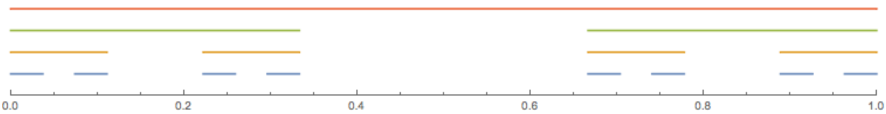

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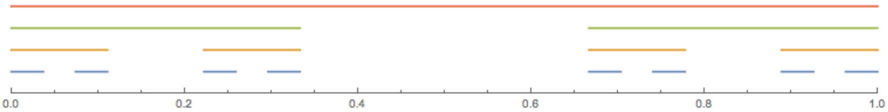

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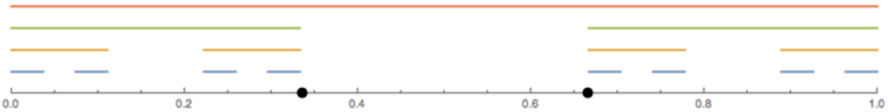

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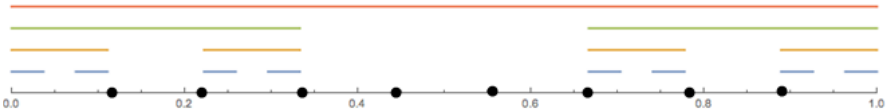

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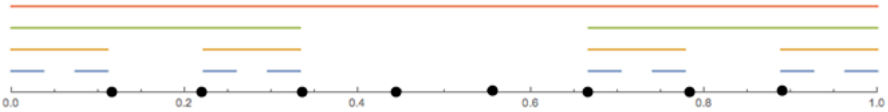

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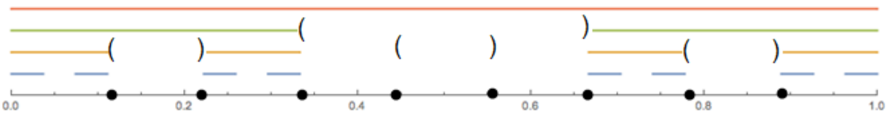

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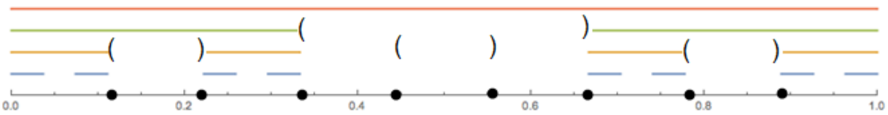

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- Formally, the Cantor set C_1 is given by

$$C_1 = \{x \in [0,1] : \forall k \; x \text{ is not in a } k\text{-gap}\}.$$

The Sierpiński Carpet

- A k-**gap** is an open interval in $[0, 1]$ of the form $(\frac{i}{3^k}, \frac{i+1}{3^k})$ with $i \equiv 1 \pmod 3$. A n-dimensional k-**pocket** is any set of the form $\gamma_1 \times \cdots \times \gamma_n$ for k-gaps γ_i.

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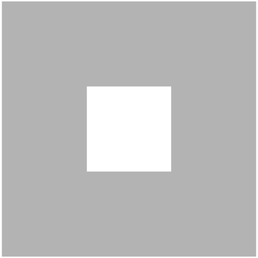

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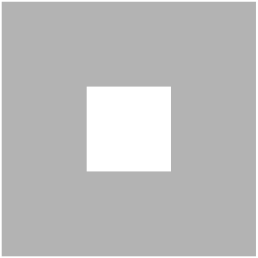

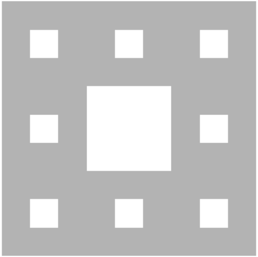

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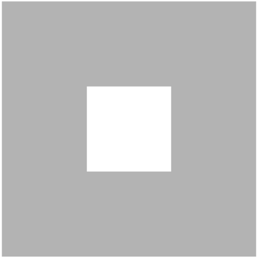
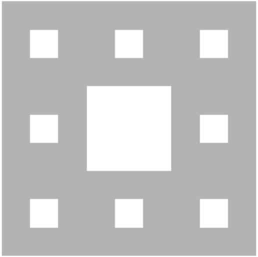
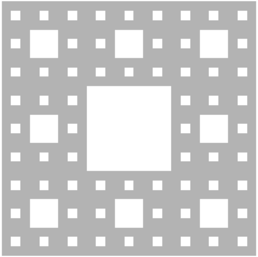

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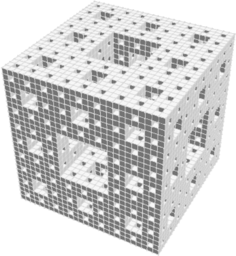
- There exists a family of higher-dimensional generalizations of C_1 with similar properties.



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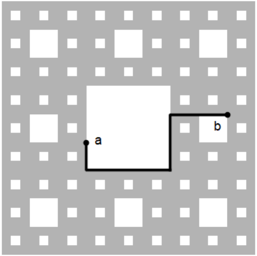

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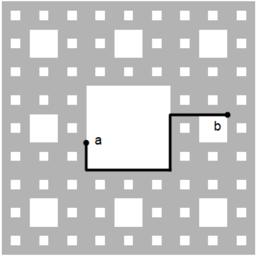

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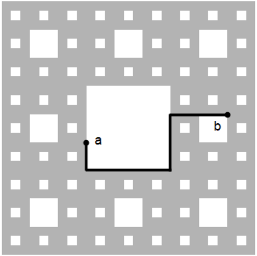

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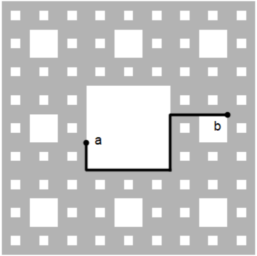

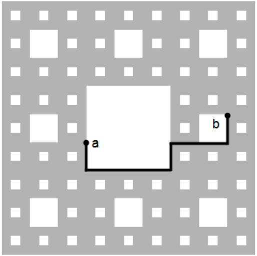

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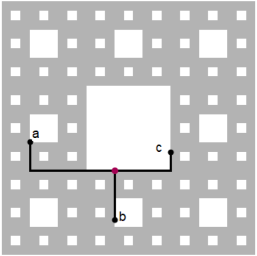

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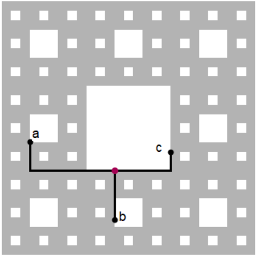

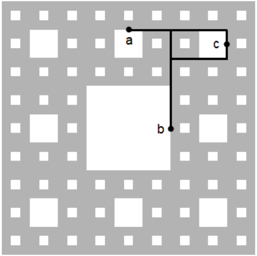

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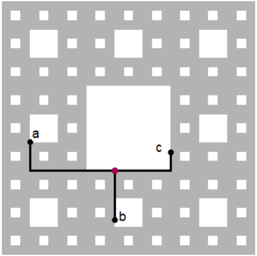

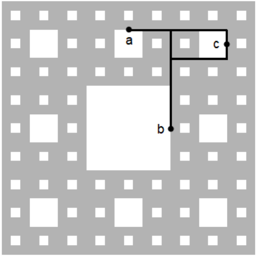

- Can we characterize the existence of a triple intersection point in C_2?

Mesh Construction

- Given $a, b \in C_2$ we consider all shortest paths between a and b at once via the **mesh of shortest paths from a to b**,

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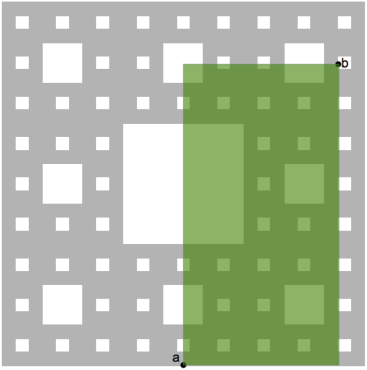

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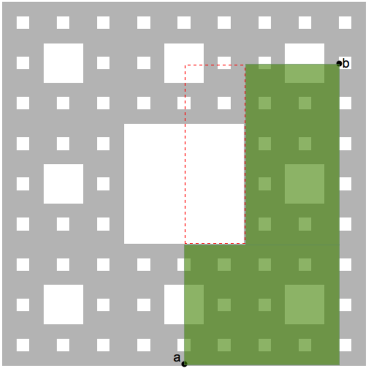

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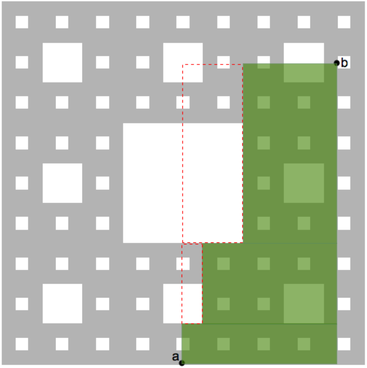

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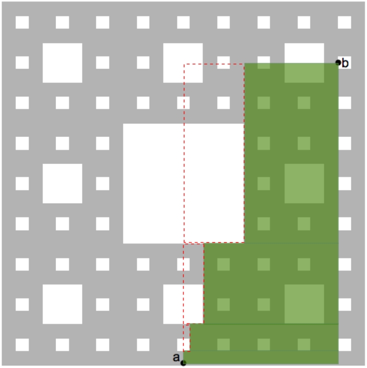

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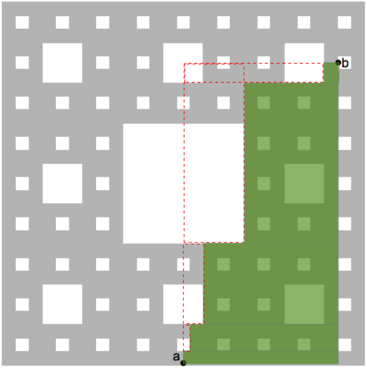

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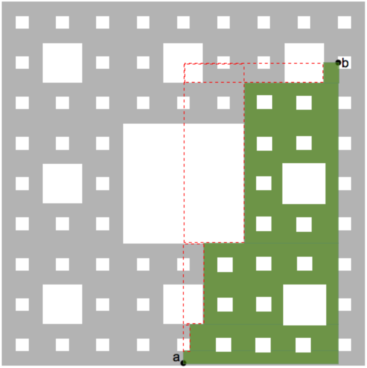

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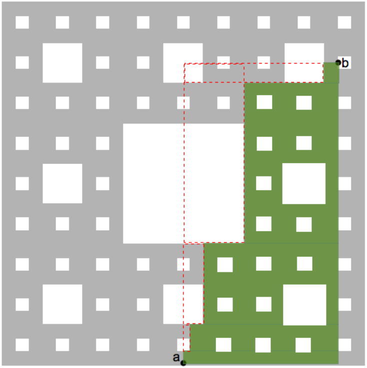

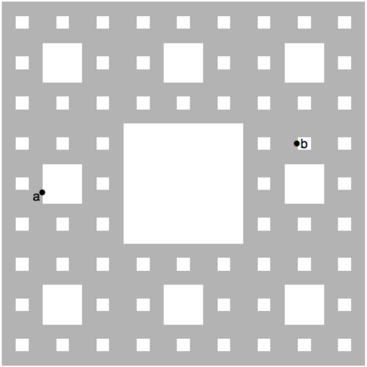

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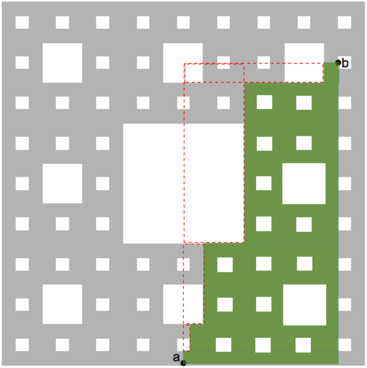

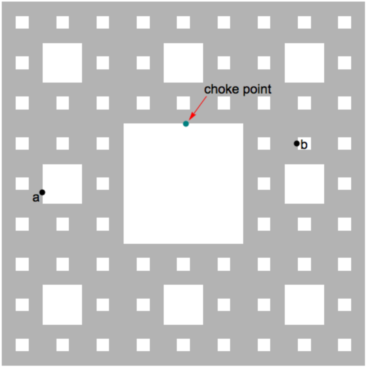

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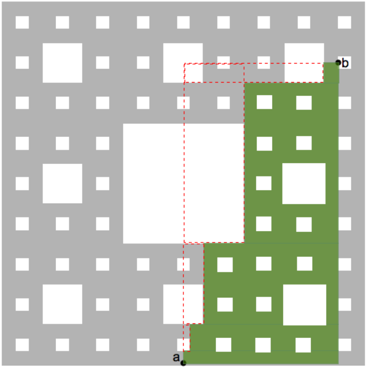

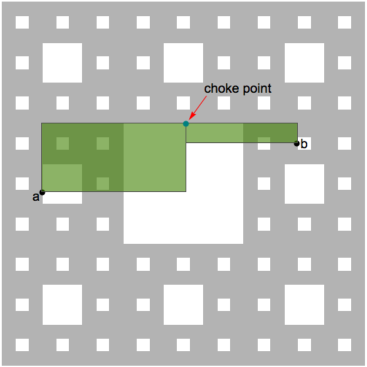

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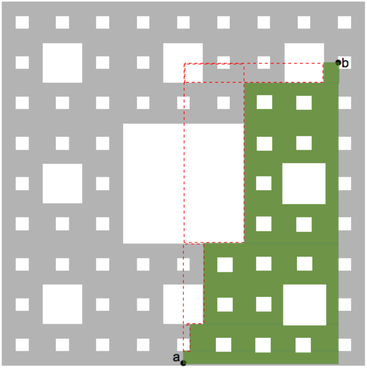

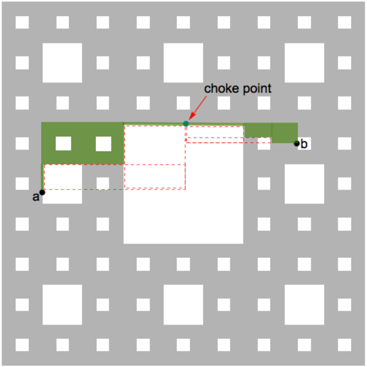

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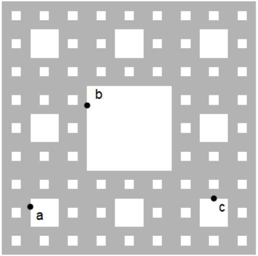

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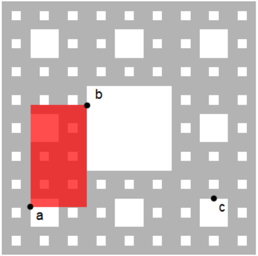

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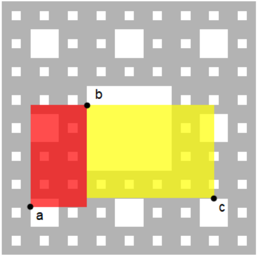

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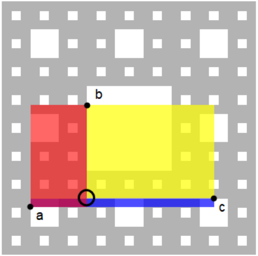

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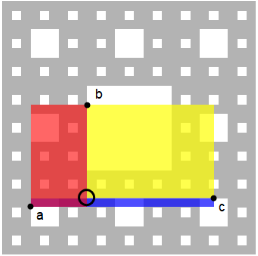

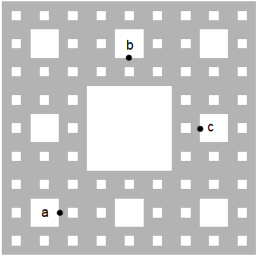

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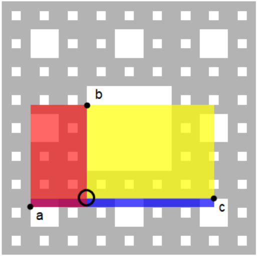

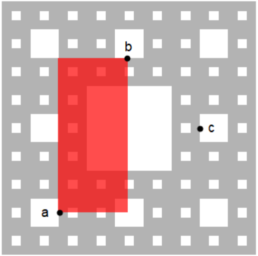

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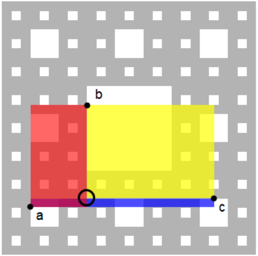

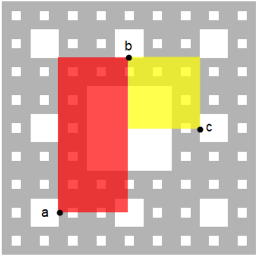

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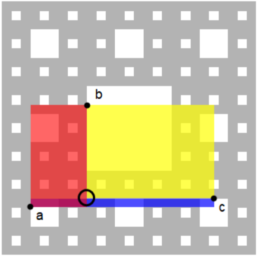

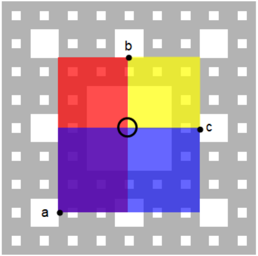

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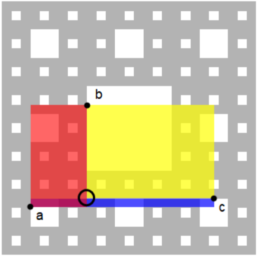

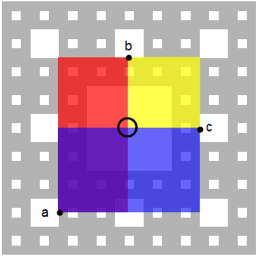

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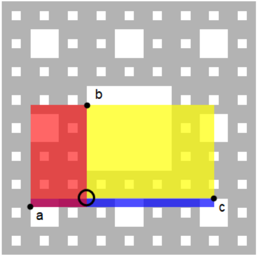

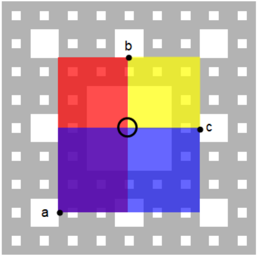

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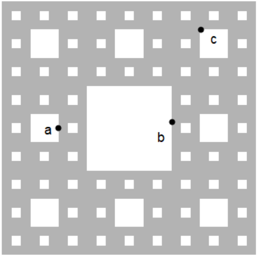

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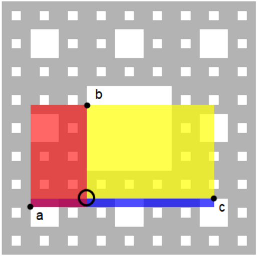

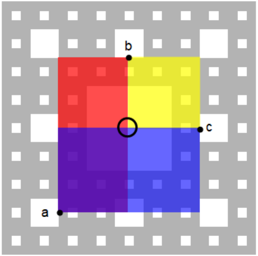

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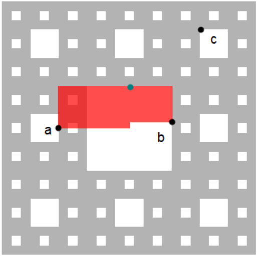

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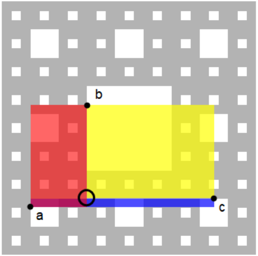

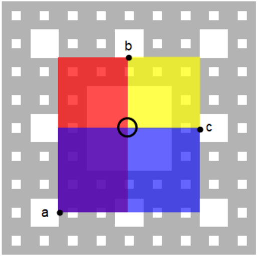

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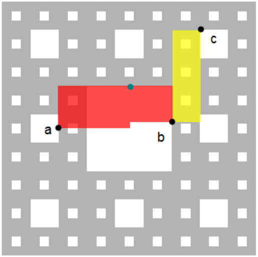

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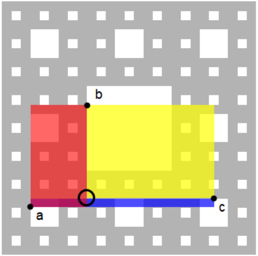

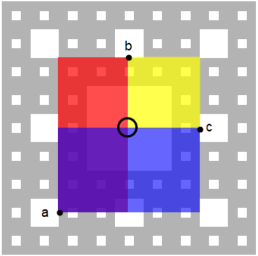

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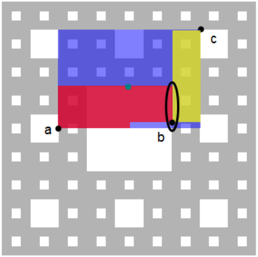

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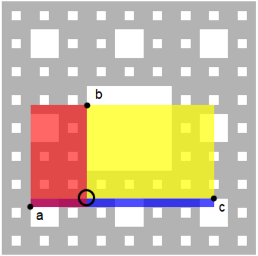

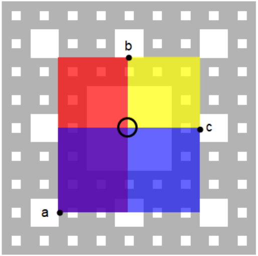

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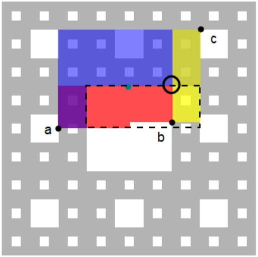

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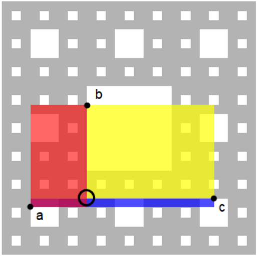

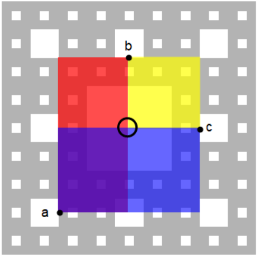

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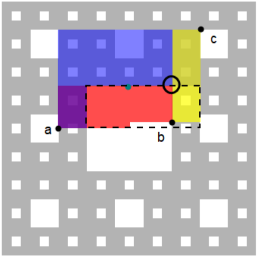

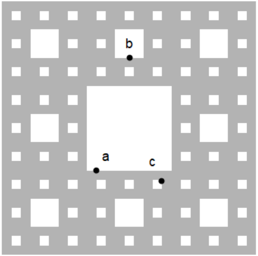

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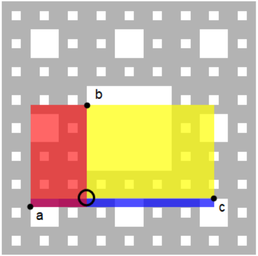

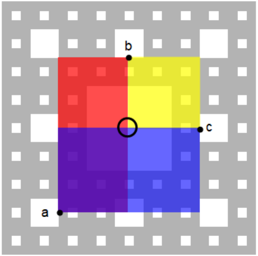

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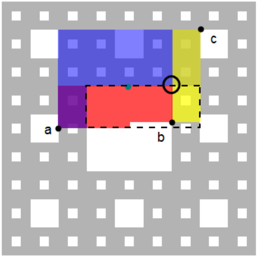

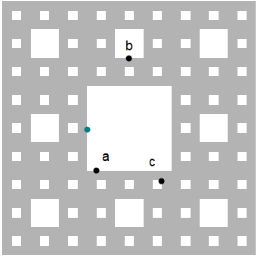

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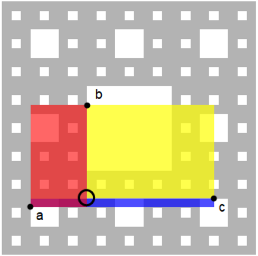

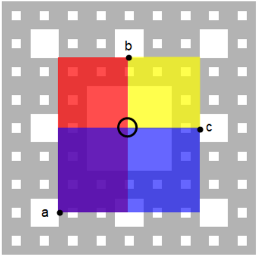

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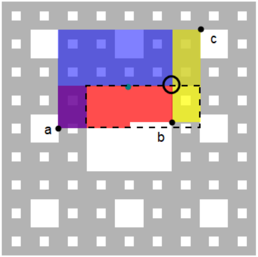

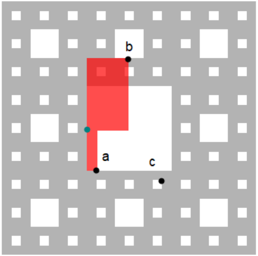

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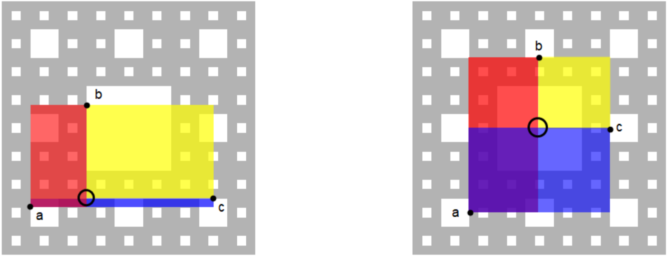

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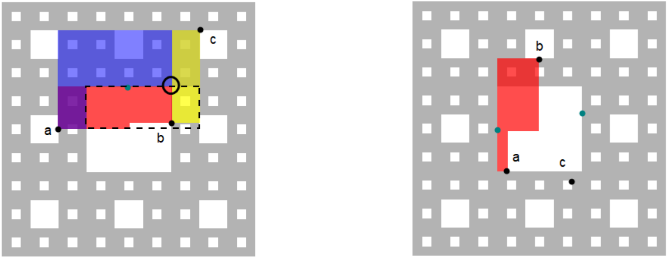

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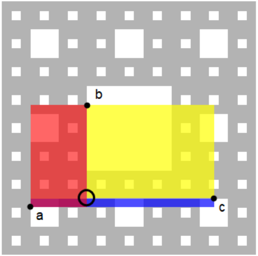

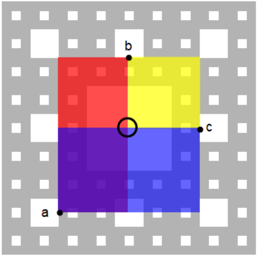

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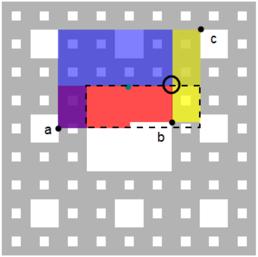

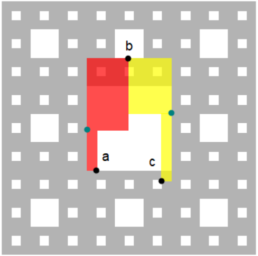

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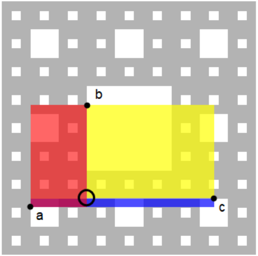

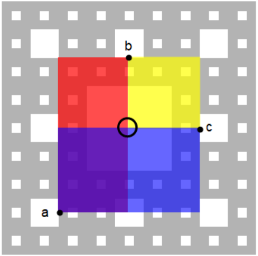

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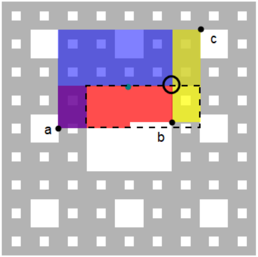

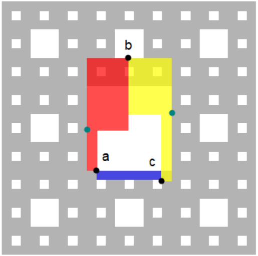

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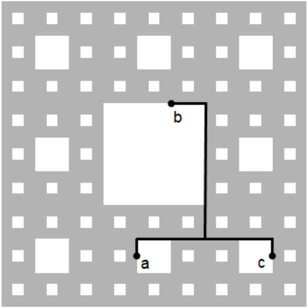

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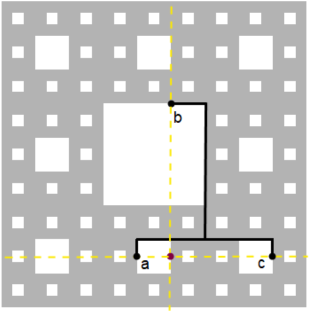

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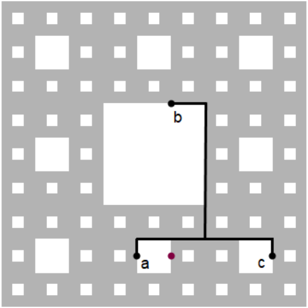

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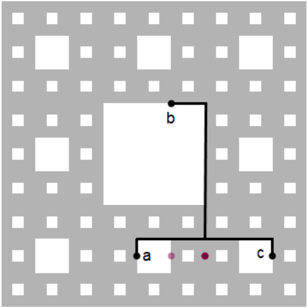

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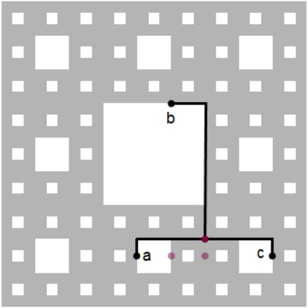

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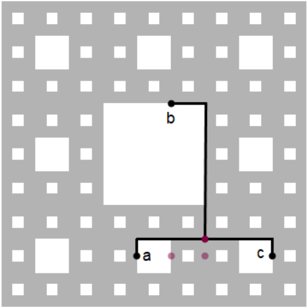

Theorem

Given $a, b, c \in C_2$, let $M = M_{ab} \cap M_{bc} \cap M_{ca}$. If M is nonempty, M contains only the rounded median.

Results

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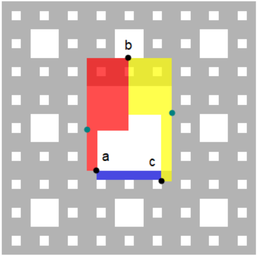

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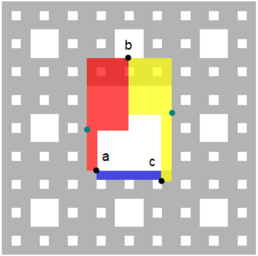

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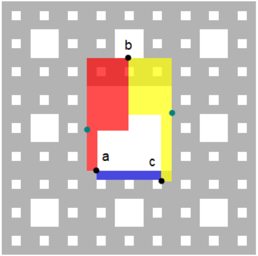

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Theorem

M is nonempty if and only if all detours of pairwise shortest paths are compatible and there is no detour on any shortest path from a, b or c to the rounded median.

Future Directions: Higher Dimensions

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- Thank you!