One-Seventh Ellipse Problem

Sanah Suri

Grinnell College
27^{th} January 2018

What is the One-Seventh Ellipse problem?

- $\frac{1}{7} = 0.142857142857\cdots$
  $= 0.\overline{142857}$

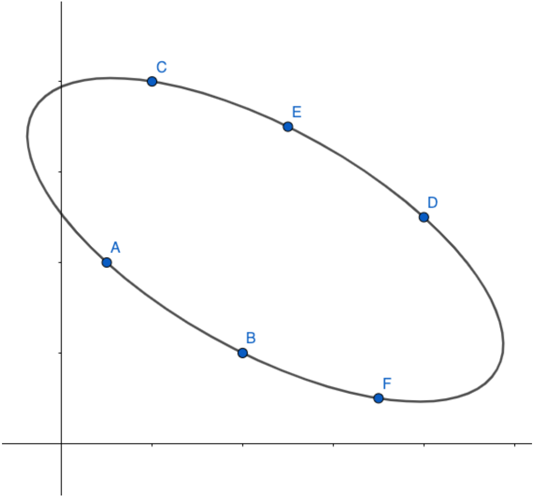
Repeating sequence: $1, 4, 2, 8, 5, 7$

Select a set of six points based on the above sequence

$\{(1,4),(4,2),(2,8),(8,5),(5,7),(7,1)\}$

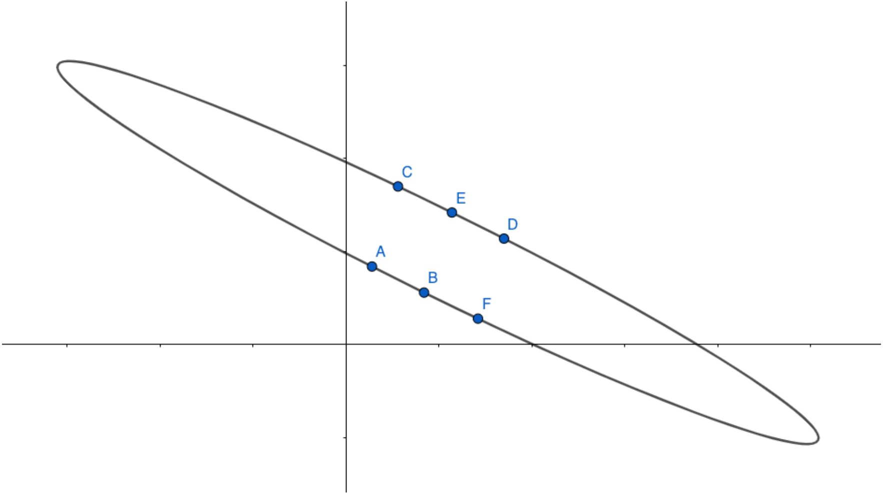
What is the One-Seventh Ellipse problem?

We have the points $(1,4),(4,2),(2,8),(8,5),(5,7),(7,1)$ lying on the following ellipse



What is the One-Seventh Ellipse problem?

Interestingly, the points $(14, 42), (42, 28), (28, 85), (85, 57), (57, 71)$ also lie on an ellipse:



Why is it interesting?

- We already know that 5 points determine an ellipse. What about the sixth point?
- Is $\frac{1}{7}$ a special case?

Observations

$1, 4, 2, 8, 5, 7$

Let's consider more sets of points:

$P_1 = \{(1, 4), (4, 2), (2, 8), (8, 5), (5, 7), (7, 1)\}$

$P_2 = \{(1, 2), (4, 8), (2, 5), (8, 7), (5, 1), (7, 4)\}$

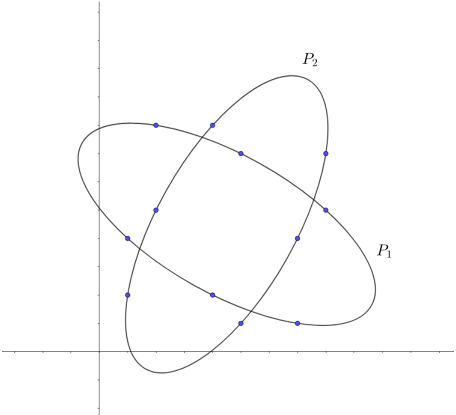
Observations

$1, 4, 2, 8, 5, 7$

Let's consider more sets of points:

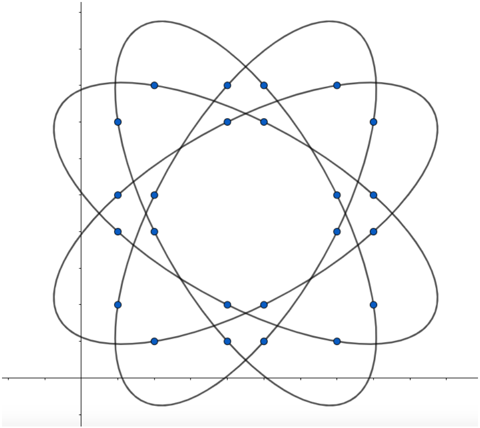
$P_1 = \{(1,4), (4,2), (2,8), (8,5), (5,7), (7,1)\}$

$P_2 = \{(1,2), (4,8), (2,5), (8,7), (5,1), (7,4)\}$



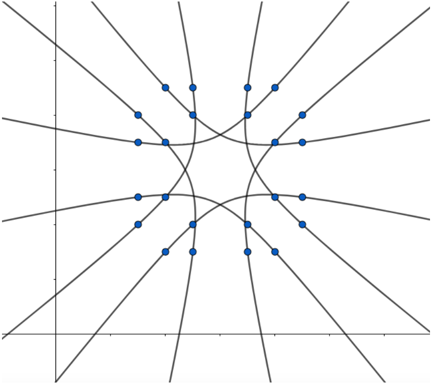
Observations

In fact:



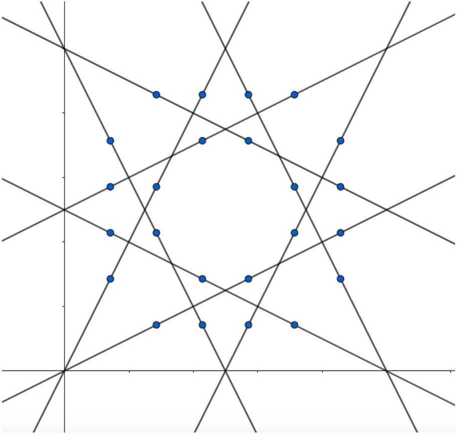
Observations

Another example is the sequence $5, 4, 3, 7, 8, 9$



Observations

And $142, 428, 285, 857, 571, 714$



Generalization

Our findings:

Generalization

Our findings:

- The fraction $\frac{1}{7}$ and the resulting sequence is not a special case

Generalization

Our findings:

- The fraction $\frac{1}{7}$ and the resulting sequence is not a special case
- The theorem can be generalized to any sequence of six numbers that hold a certain property

Generalization

Our findings:

- The fraction $\frac{1}{7}$ and the resulting sequence is not a special case
- The theorem can be generalized to any sequence of six numbers that hold a certain property
- We can extend this to all conic sections

Generalization

Notice that in the sequence $1, 4, 2, 8, 5, 7$, we have

$$\begin{gathered} 1, 4, 2 \\ 8, 5, 7 \end{gathered}$$

$$1 + 8 = 4 + 5 = 2 + 7 = 9$$

Similarly,

$$5 + 7 = 4 + 8 = 3 + 9 = 12$$

Generalization

We can generalize the sequence to

$$a, \quad b, \quad c, \quad S-a, \quad S-b, \quad S-c$$

How do we construct the six points?

Generalization

We can generalize the sequence to

$$a, \quad b, \quad c, \quad S - a, \quad S - b, \quad S - c$$

How do we construct the six points?

Let $n \in \{0, 1, 2, 3, 4, 5\}$

- x-coordinate: i^{th} entry
- y-coordinate: $(i + n)^{th}$ entry (wraps around if we hit the end)

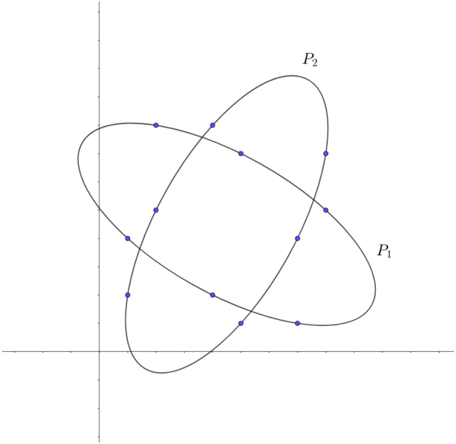
Call the set of these six points P_n corresponding to the chosen n

Generalization

$1, 4, 2, 8, 5, 7$

$P_1 = \{(1,4), (4,2), (2,8), (8,5), (5,7), (7,1)\}$

$P_2 = \{(1,2), (4,8), (2,5), (8,7), (5,1), (7,4)\}$



Theorem

Suppose $a, b, c, S - a, S - b, S - c$ are six distinct real numbers. For each $n \in \{0, 1, 2, 3, 4, 5\}$, we have the following properties:

Theorem

Suppose $a, b, c, S-a, S-b, S-c$ are six distinct real numbers. For each $n \in \{0, 1, 2, 3, 4, 5\}$, we have the following properties:

1. All elements of P_n lie on a unique conic section, which is degenerate (straight lines) when $n = 0$ and $n = 3$.

Part 1

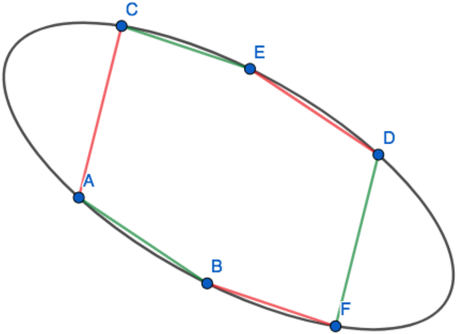
All elements of P_n lie on a unique conic section, which is degenerate (straight lines) when $n = 0$ and $n = 3$.

Part 1

Braikenridge-Maclaurin Theorem: If three lines meet three other lines in nine points and three of these points are collinear, then the remaining six points lie on a conic section.

Part 1

Braikenridge-Maclaurin Theorem: If three lines meet three other lines in nine points and three of these points are collinear, then the remaining six points lie on a conic section.



Part 1

- Why is the conic unique: five points determine a conic.
- $P_0 = \{(1,1),(4,4),(2,2),(8,8),(5,5),(7,7)\}$ and $P_3 = \{(1,7),(4,5),(2,8),(8,2),(5,4),(7,1)\}$ obviously lie on $x = y$ and $x + y = S$ respectively.

Theorem

Suppose $a, b, c, S-a, S-b, S-c$ are six distinct real numbers. For each $n \in \{0, 1, 2, 3, 4, 5\}$, we have the following properties:

1. All elements of P_n lie on a unique conic section, which is degenerate (straight lines) when $n = 0$ and $n = 3$.

Theorem

Suppose $a, b, c, S-a, S-b, S-c$ are six distinct real numbers. For each $n \in \{0, 1, 2, 3, 4, 5\}$, we have the following properties:

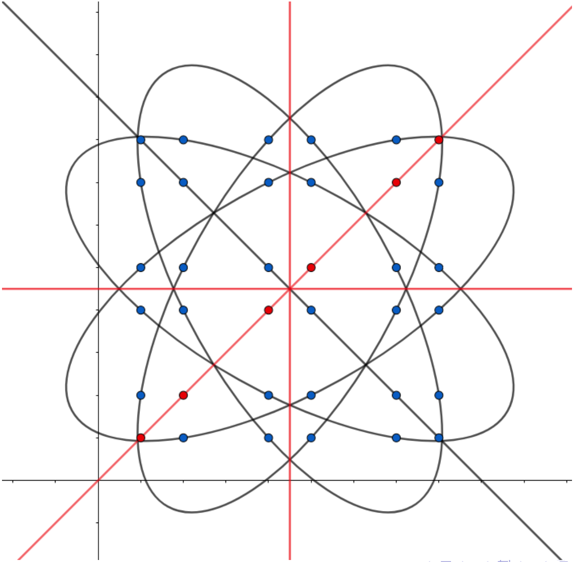
1. All elements of P_n lie on a unique conic section, which is degenerate (straight lines) when $n = 0$ and $n = 3$.
2. If $n \neq n'$, then both $P_{n'}$ (and its associated conic) can be obtained by appropriately reflecting the points in P_n (and its associated conic).

Part 2

If $n \neq n'$, then both $P_{n'}$ (and its associated conic) can be obtained by appropriately reflecting the points in P_n (and its associated conic).

Part 2

If $n \neq n'$, then both $P_{n'}$ (and its associated conic) can be obtained by appropriately reflecting the points in P_n (and its associated conic).



Part 2

Lines of reflection:

- $x = y$
- $x = \frac{S}{2}$
- $y = \frac{S}{2}$
- $x + y = S$

Further Observations

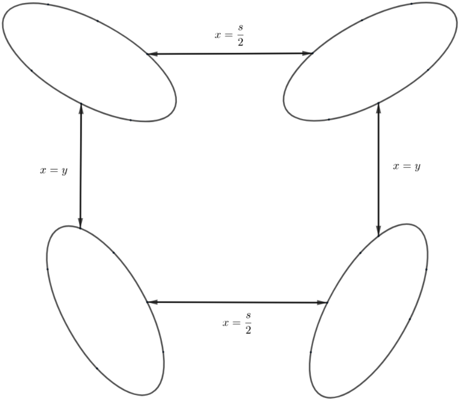
The reflection of the ellipses has a group structure isomorphic to D_2

$$D_2 = \{e, s, t, st\}$$

Further Observations

The reflection of the ellipses has a group structure isomorphic to D_2

$$D_2 = \{e, s, t, st\}$$



Future Goals

- Is there a geometric way of telling what kind of conic section will be formed using the points?
- Can this be extended to longer sequences of numbers?
- Is this possible in a higher dimension?

Acknowledgements

I would like to thank my research collaborator, Shida Jing, our research mentor, Professor Marc Chamberland and the organizers and volunteers at NCUWM!

Thank you for listening!

My email:
surisana@grinnell.edu