Smooth Stable Distributions and the Central Limit Theorem

School of Mathematical and Statistical Sciences
Arizona State University

An examination of the Generalized Central Limit Theorem and exploration of stable distributions in general.

Liliaokeawawa (Lily) Cothren, Arizona State University, 623.261.9828, lcothren@asu.edu

What are Stable Distributions?

Definition: A random variable X is stable or stable in the broad sense if for independent copies of X, X_1 and X_2, and any positive real constants *a* and *b*,
$a\,X_1 + b\,X_2 = cX + d$ holds for some positive real constants *c* and *d* [3].
Although analytically inexpressible, stable distributions' features can be estimated by the following parameters:
Index of stability: measures probability of data in the extreme tails, and is denoted by α, ranging within (0, 2). A normal distribution has α = 2. Smaller values of α result in heavier tails.
Skewness parameter: This is usually denoted by β, ranging within [-1, 1]. If β = 0, then the distribution is symmetrical.
Scale parameter: Usually denoted by γ, this measures dispersion of data. γ is half the population variance for normal distributions.
Location parameter: δ is equal to the median. For the normal distribution, the mean is a good indicator for the value of δ [3].

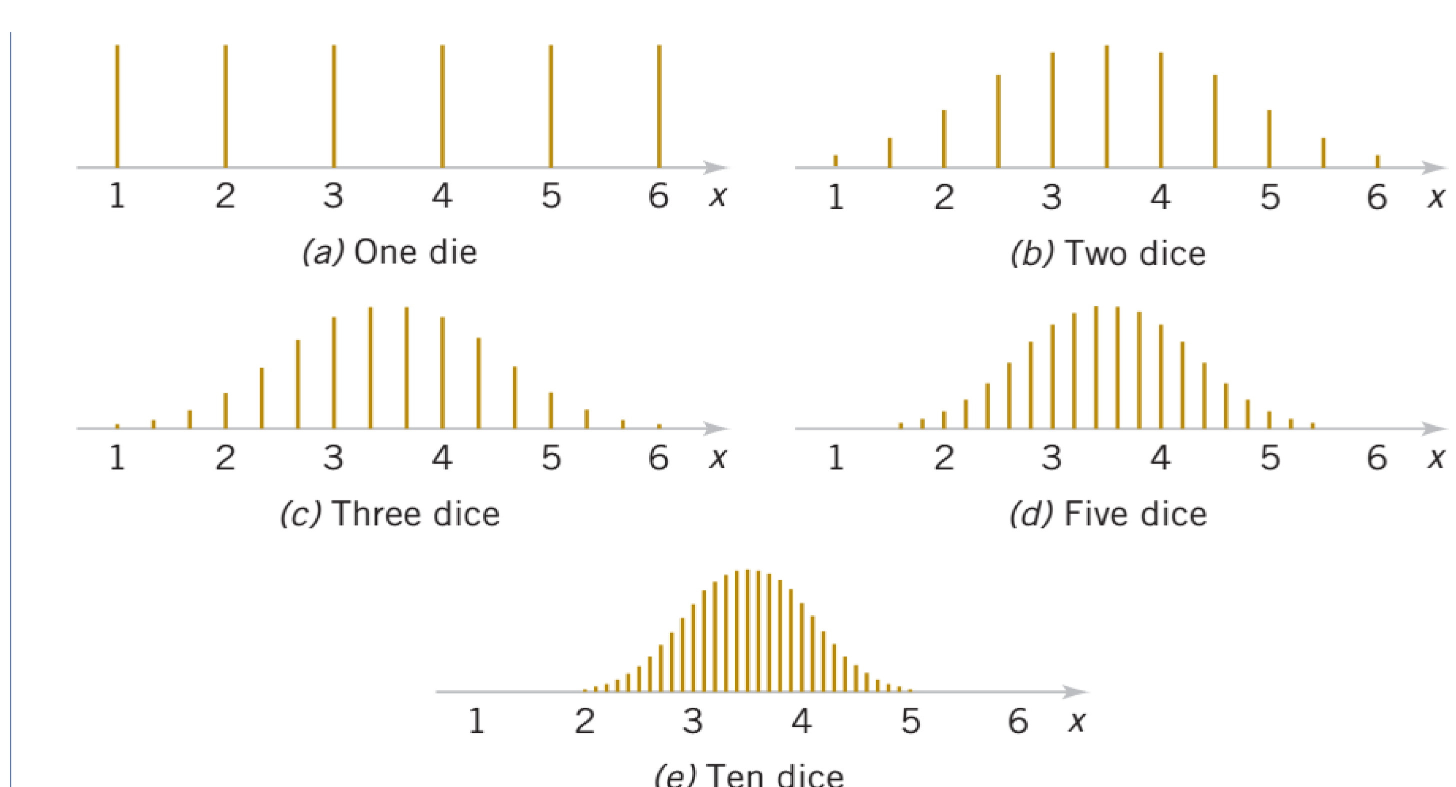
Common Examples

Although most stable distributions cannot be written in closed form, a few popular distributions exist [3]:

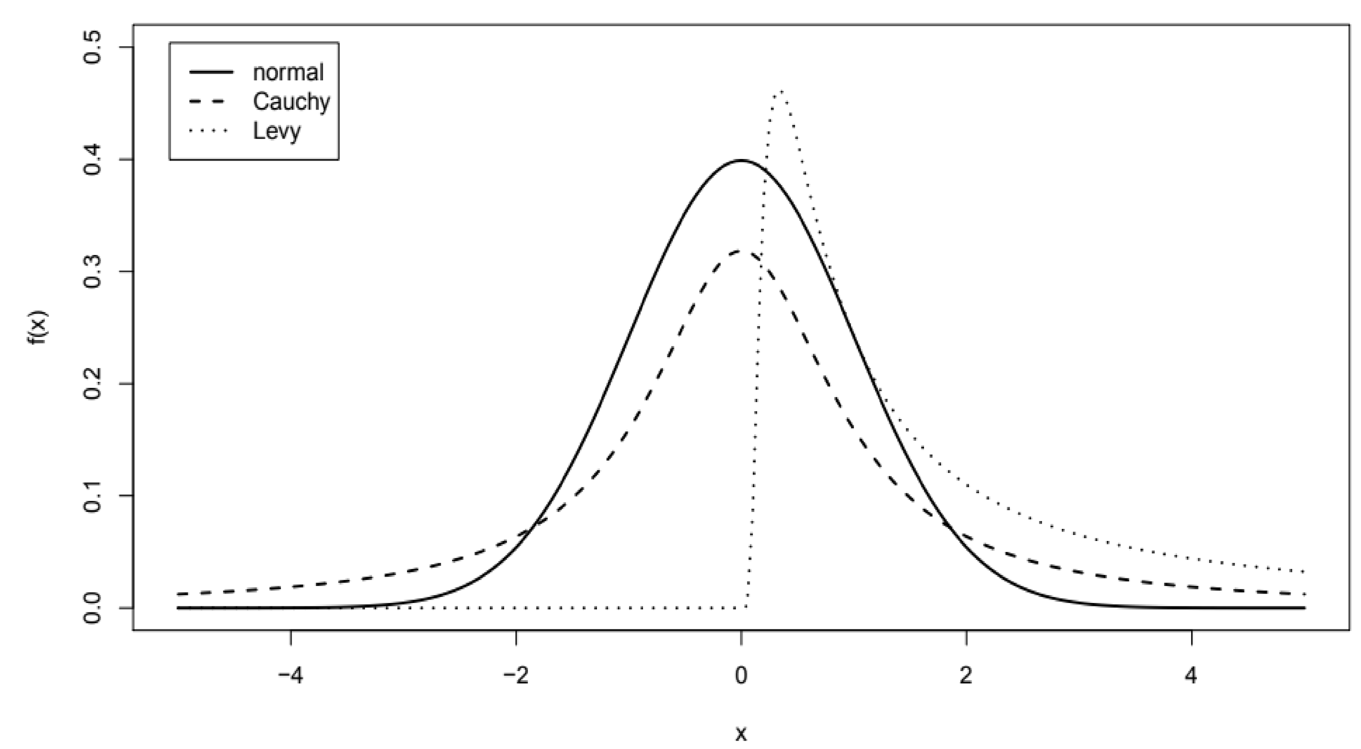
- Normal or Gaussian Distribution, X ~ N(μ, σ^2)
  - $f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), -\infty < x < \infty$
- Cauchy Distribution, X ~ Cauchy(γ, δ)
  - $g(x) = \left(\frac{1}{\pi}\right)\frac{\gamma}{\gamma^2+(x-\delta)^2}, -\infty < x < \infty$
- Levy Distribution, X ~ Levy(γ, δ)
  - $h(x) = \sqrt{\frac{\gamma}{2\pi}}\frac{1}{(x-\delta)^{\frac{3}{2}}}\exp\left(-\frac{\gamma}{2(x-\delta)}\right), \delta < x < \infty$

Extended Properties

- All (non-degenerate) stable distributions are continuous distributions with an infinitely differentiable density [1].
- All stable distributions are infinitely divisible [1].
- All stable distributions except the normal distribution are leptokurtotic and heavy-tailed [3].
- All stable distributions are closed under convolution for fixed values of α and β [3].
- Any moment greater than α isn't defined, meaning any question involving the second moment (variance) isn't useful [3].



Above is a graphic explaining the Central Limit Theorem pictorially [2].



Pictured above are the three stable distributions with closed form: Gaussian, Cauchy, and Levy [3].

Stable Distributions are Smooth: Proof Outline and Further Investigation

Few approaches exist to prove that stable distributions are indeed continuous and infinitely differentiable. Currently, collaborators on this independent research project are still in the throws of connecting this proof to that of a Generalized Central Limit Theorem, or framing characteristic functions - of which there are many varieties - to fit that of a "differential equations" perspective. Below are proof outlines the collaborators of this project plan to pursue in greater detail:
First, identify characteristic equations in question which are useful:
Given a random variable X = aZ + b, if

$$X = \begin{cases} \gamma Z + \delta, \alpha \neq 1 \\ \gamma Z + (\delta + \beta\left(\frac{2}{\pi}\right)\gamma log\gamma, \alpha = 1 \end{cases}$$

And $signu = \begin{cases} -1, u < 0 \\ 0, u = 0 \\ 1, u > 0 \end{cases}$ and where $Z = Z(\propto, \beta)$ is a random variable with characteristic function is given by:

$$Eexp(iuZ) = \begin{cases} \exp\left(-|u|^\alpha\left[1 - i\beta\left(\tan\left(\frac{\pi\alpha}{2}\right)\right)(signu)\right]\right), \alpha \neq 1 \\ \exp\left(-|u|\left[1 - \frac{i\beta 2}{\pi}(signu)\log|u|\right]\right), \alpha = 1 \end{cases}$$

X then has the useable characteristic function:

$$Eexp(iuX) = \begin{cases} \exp\left(-\gamma^\alpha|u|^\alpha\left[1 - i\beta\left(\tan\left(\frac{\pi\alpha}{2}\right)\right)(signu)\right] + i\delta u\right), \alpha \neq 1 \\ \exp\left(-\gamma|u|\left[1 - \frac{i\beta 2}{\pi}(signu)\log|u|\right] + i\delta u\right), \alpha = 1 \end{cases}$$

Second, assert Lipschitz-ness and use contraction mapping (with a possible nod to the Picard-Lindelof Theorem) to arrive at a proof that all non-degenerate stable distributions are smooth.

Generalized Central Limit Theorem

The Central Limit Theorem (CLT) theorizes that when numerous independent random variables are linearly combined that their sum tends to converge to a normal/Gaussian distribution [2]. While this result is useful and good enough for most applications, we forget the bigger, more interesting picture that mathematicians Gnedenko and Kolmogorov set for us: the sum of these independent, identically distributed curves simply is a stable distribution [3]. Hence, a re-examination of the CLT more generally is requested later in this project [4].

Conclusion: Next Steps and Uses

Several authors have commented on the superior use of stable distributions in applications, including in finance and communications. For example, Professor of Economics J. Huston McCulloch at New York University explains that since "financial asset returns are the cumulative outcome of a vast number of pieces of information and individual decisions arriving continuously in time", then asset returns may make use of the Central Limit Theorem because the output data is the result of an additive process [4]. Moreover, though, because these types of data are often more leptokurtic (meaning to have sharp, distinctive peaks and fatter tails) than normative (having shape similar to a normal curve), statisticians argue that a stable distribution rather than a Gaussian distribution be used to model additive data since stable distributions have overall heavier tales [4]. Consequently, stable distributions are cited as a result of the Generalized Central Limit Theorem, which theorizes that the sum of independent and identically distributed densities is a stable distribution [3]. Several proofs and discussions exist on the topic, but little literature discusses other impacts that stable distributions can enact. For example, earlier we mentioned in the "Extended Properties" section that stable distributions are smooth [1]. Though this wasn't nearly as dazzling as the ever popular Generalized Central Limit Theorem, such a property raises questions about the use of stable distributions in differential equation applications. Is there room for a new perspective, such as through the lens of a functional analyst?

References (used on this poster):

[1] Chow, Yuan Shih, and Henry Teicher. *Probability Theory: Independence, Interchangeability, Martingales*. Third. New York, NY: Springer Texts in Statistics, n.d. Accessed January 20, 2020.
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[3] Nolan, John. "Basic Properties of Univariate Stable Distributions." In *Stable Distributions Models for Heavy Tailed Data*, 34. Math/Stat Department, American University: American University, 2018.
[4] Peng, Shige. "A Generalized Central Limit Theorem with Applications to Econometrics and Finance." *Institute of Mathematics, Columbia University*, Seminar of Econometrics, November 25, 2008, 33.