Exploration of Julia sets arising from centered polygonal Lacunary functions

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Introduction to Lacunary Functions

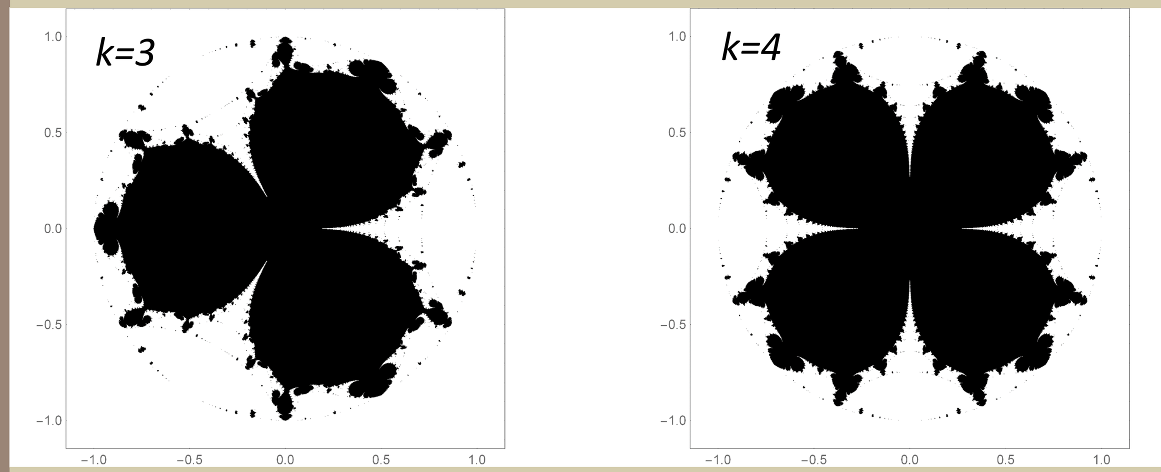
Lacunary functions are in the form of $f(z) = \sum_{n=1}^{\infty} z^{g(n)}$, where $g(n)$ has "gaps" in its values. Take for example $f(z) = \sum_{n=1}^{\infty} z^{n^2}$. The infinite series would be

$$z + z^4 + z^9 \dots z^{\infty}.$$

If the "gaps" in powers are ever increasing then a *natural boundary*, a dense curve of singularities, will form. Our work will focus on lacunary functions called centered polygonal lacunary functions. The $g(n)$ will be a centered polygonal number, which is

$$C(k) = \frac{kn^2 - kn + 2}{2} \quad k, n \in \mathbb{N}^+.$$

We are interested in filled-in Julia sets arising from the centered polygonal lacunary functions. A filled-in Julia set is created by iterating a function an infinite amount of times. In this case, we will be plugging in $f(z)$ for z, $f(f(z))$. As we do this an infinite amount of times, $f\left(f\left(f(z \dots f(z))\right)\right)$, we will produce a fractal called a filled-in Julia set. The filled-in Julia sets for $k = 3$, and $k = 4$ are shown below. A black point means the point converges to 0. And a white point means it diverges to infinity.



Applications

Lacunary functions have several applications in Physics. Natural boundaries have been found to have impact on quantum tunneling [1] and evanescent waves outside of elliptic dielectrics [2]. Due to the relation to Weiner (stochastic) processes, natural boundaries have been discussed in the context of Brownian motion [3].

Filled-in Julia Sets

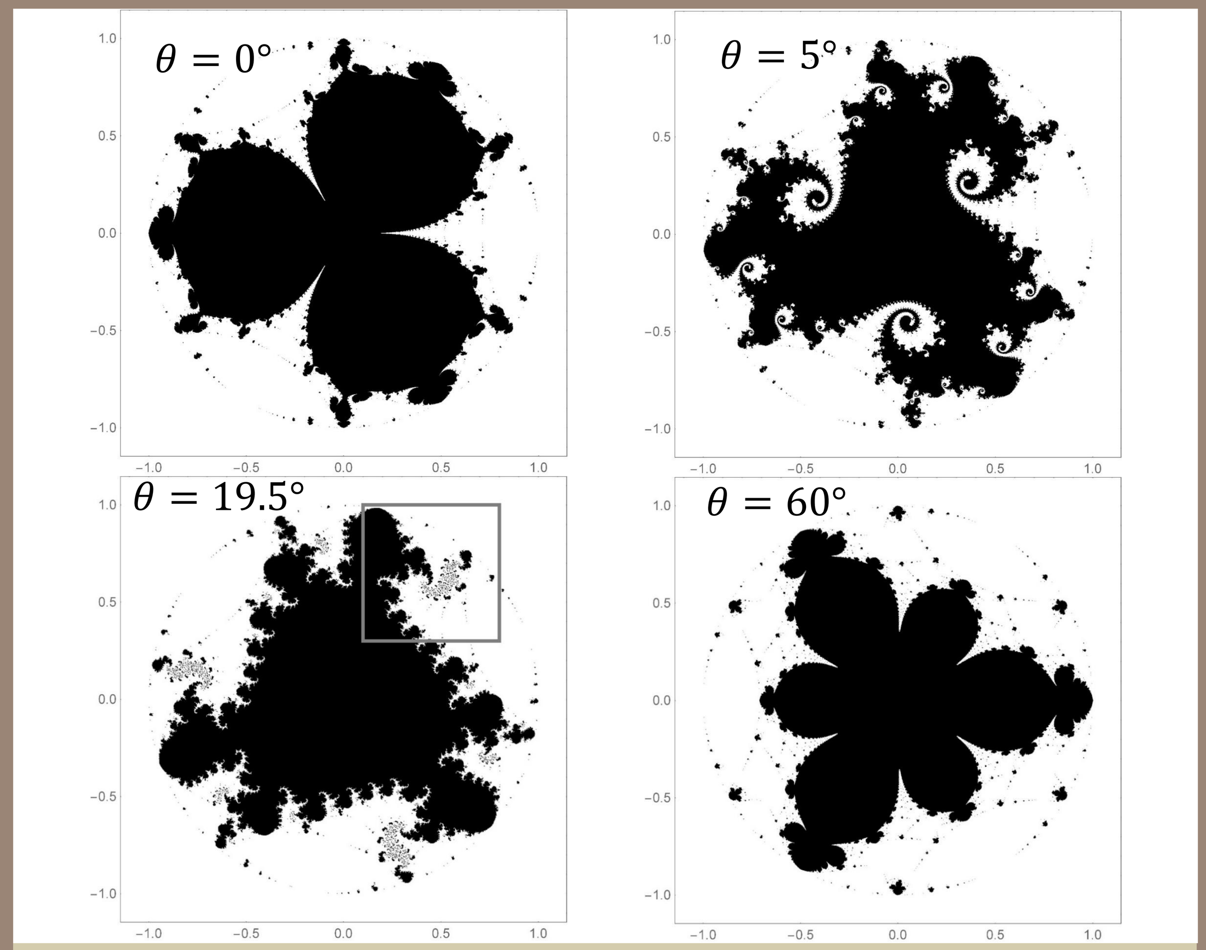
The filled-in Julia sets for these centered polygonal lacunary functions are shown above and to the right. There are a few things of note:

- k-fold symmetry
- Symmetry about the real axis
- k number of distinct lobes
- Strings of groups of non-diverging points spanning the cleft between lobes

Of course, for $f(z) = \sum_{n=1}^{\infty} z^{g(n)}$, z doesn't have to be simple, one can add a phase rotation component and a complex offset shift component. This will look like $f(z) = \sum_{n=1}^{\infty} (e^{i\theta} z + c)^{C(k)}$. This will have the $e^{i\theta}$ term as the phase rotation. This is done by changing θ. The c term will be some complex number, and this is the offset component. First, the phase rotation information will be presented.

References

[1] Shudo, A., Ikeda, K.S. *Phys. Rev. Lett.* **109**, 154102 (2012)
[2] Creagh, S.C., White, M.M.: *J. Phys. A* **43**, 465102 (2010)
[3] Hille, E.: *Analytic function theory Vol. II*, Ginn and Company, Boston, MA (1962)



Exploring Phase Rotation

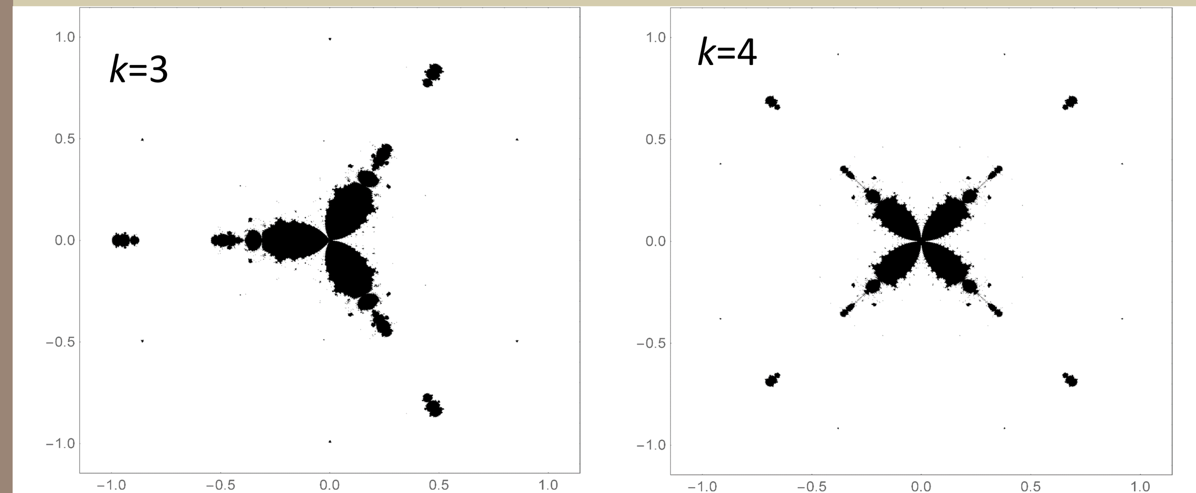
Changing the θ will produce new filled-in Julia sets that can be unique from the *base filled-in Julia set* ($\theta = 0°$). Attention is be given to the range of θ values, $0 \leq \theta \leq \frac{360°}{k}$.

All the images above are for $k = 3$. The upper left graph shows the base filled-in Julia set, $\theta = 0°$. Of note is the sharp cleft pointing toward the origin. The upper right graph shows the filled-in Julia set for $\theta = 5°$, this graph shows the spirals that form as phase rotation is induced. Although this set has lost its symmetry about the real axis, it has maintained its rotational symmetry.

The bottom left graph shows the filled-in Julia set for $\theta = 19.5°$. Again the spirals can be noted, but more important is the region in the gray box. This is what we call an *unstable island*. As one can see, the unstable island is only seen in the bottom left graph, this is because the unstable island is forming into one solid island as shown in the lower graph, and will dissolve at about $\theta = 25°$.

Mandelbrot Sets

A Mandelbrot set is formed by changing the offset shift term, c. The origin is then calculated to determine whether it will converge to 0, black point, or diverge to infinity, white point. Then a range of points in the complex plane are plotted to observe trends and features.

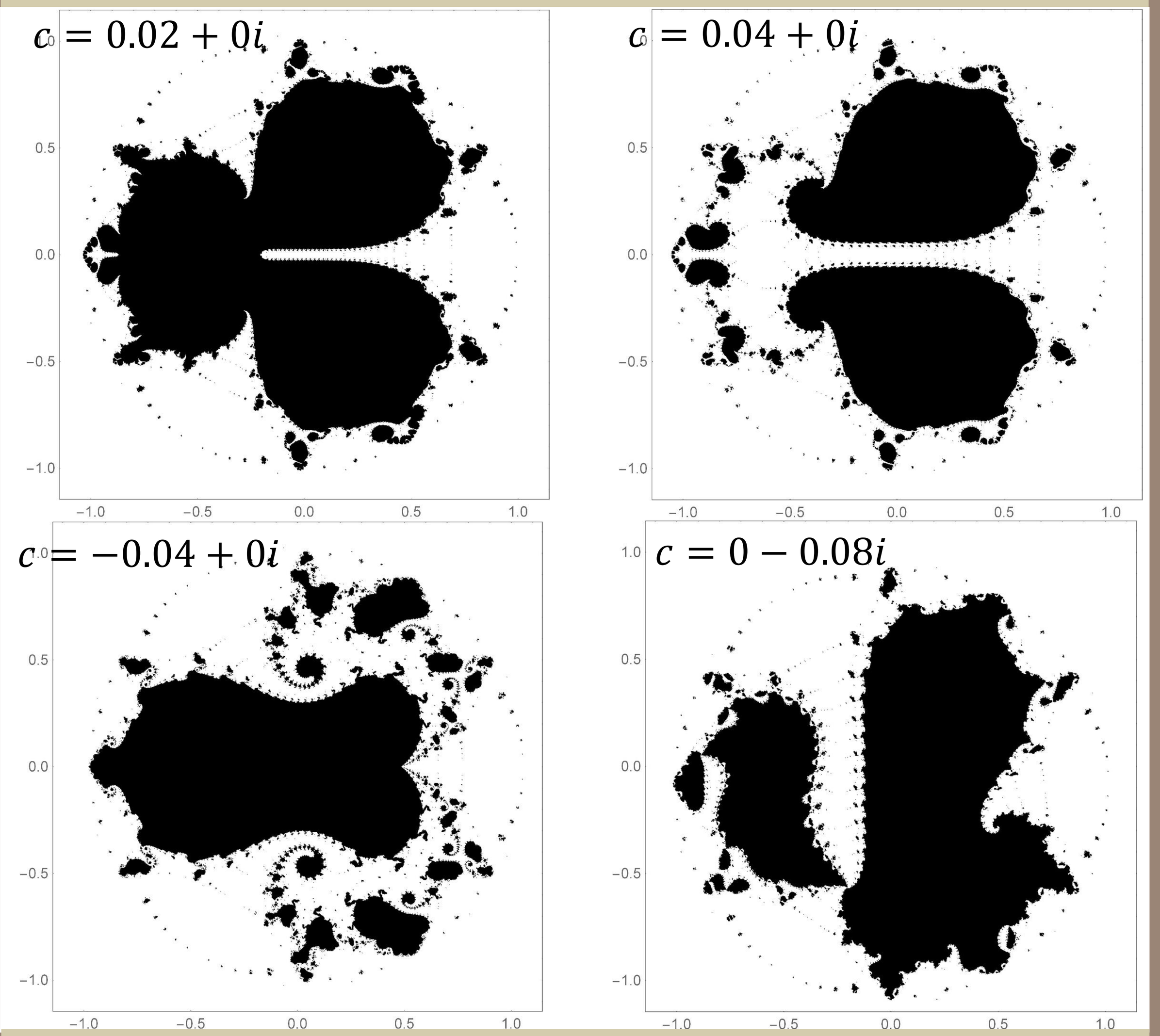


Shown above are the Mandelbrot sets for $k = 3$ and $k = 4$ respectively. One can notice several features, such as the symmetry about the real axis, the k-fold symmetry, and the k number of lobes.

Returning to the filled-in Julia sets that produce the Mandelbrot sets is integral to understanding the nature of these functions, so a few c values will be shown to observe general behavior.

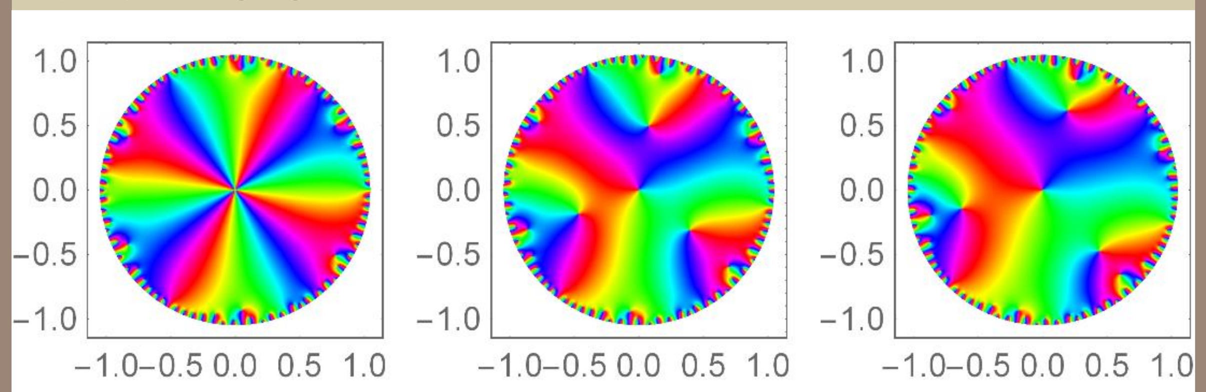
Exploring Offset Shift

The top left graph below shows the filled-in Julia set when $c = 0.02 + 0i$. There is a sharp cleft of diverging points into this filled-in Julia set. This cleft crosses the origin thus a white point will appear on the Mandelbrot set. This shows what is happening to form the deep needle of diverging points in the Mandelbrot set. Moving to the top right graph, one can see that the cleft pierces completely through the filled-in Julia set, dividing it into two halves reflected across the real axis. The bottom left graph shows a small step in the negative real direction. One can see very different behavior as there is no cleft, but instead two lobes begin to dissipate. This progression is seen as the shift in the negative direction grows larger. The bottom right graph shows the plot breaking into one oblong shape as a negative, imaginary shift is added.

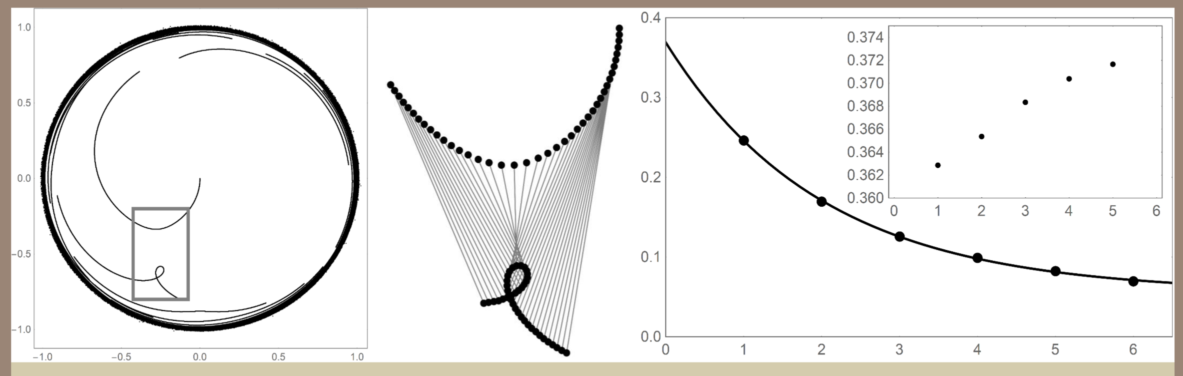


Iterative Dynamics

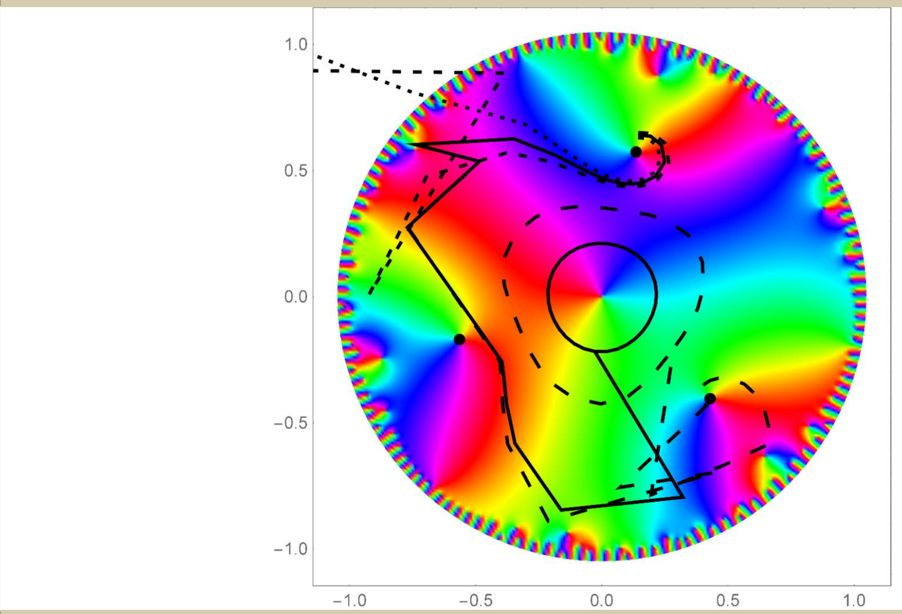
One can build a vector field to show the direction an initial point moves upon iteration of the lacunary function. Fixed points (those points that don't move) of a vector field are particularly important. In the graphs below, this can be seen as places where the color is rapidly changing.



These images are the vector fields for $k = 3$, and the θ values changes, $\theta = 0°$ on the left, $\theta = 9°$ in the center, and $\theta = 18°$ on the right. There is always a fixed point at the origin (*zero fixed point*) and many more on the edge of the unit circle (*secondary fixed points*). However, as a phase rotation constant is added, exactly k number of additional fixed points emerge from the origin. These are called *primary fixed points*. The primary fixed points will, as θ changes, quickly move from the center, then rotate and approach the natural boundary. This behavior was found for all k values.



The graphs above show the movement of fixed points for $k = 1$. The most prominent line is the primary fixed point, while the line with a loop in it is a secondary fixed point. The points in the middle graph are the location of a fixed point as θ varies and the lines connect points with the same θ value. There appears to be a "repulsion" between these two points. The graph on the left shows the minimum distance between these point as a function of k. The θ angle that the minimum distance occurs at is a weakly related function of k, as shown in the inset graph.



The above graph shows both the vector field and the paths of a few points with very similar starting values. The solid line is for the initial point $c = 0.15 + 0.64i$. After 50 steps, it enters a stable orbit. The small and medium sized dashed lines show the path of $c = 0.15 + 0.63i$ and $c = 0.15 + 0.65i$ respectively. This shows even a small change in initial position can have great effect on outcome. Altogether this demonstrates the complex nature of this lacunary function, and it great dependence on initial values.

Conclusions

Lacunary functions find many uses in Physics. This poster has shown the features of these lacunary sequences and explored much of the beautiful behavior of these sequences. The parameter space of Julia sets included a phase rotation component and an offset shift component. Qualitative features such as symmetry, unstable islands, and lobes were identified. In addition to Julia and Mandelbrot sets, related iterative dynamical maps were discussed. The nature of their fixed points and dynamic trajectories were expounded on.

It is hoped that this work will provide foundation for deeper studies into the fractal characteristics of centered polygonal lacunary functions. Specific areas of follow-up could include a more thorough investigation of the phase rotation and offset shift parameter space, and a detailed study of the nature of dynamical trajectories and their dependence on the various parameters of the centered polygonal lacunary functions.

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