Unfolding Monodromy

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Goals

This project investigates spheres and cylinders equipped with singular integral affine structures with nodes. Nodes introduce monodromy. An elementary example of monodromy is the flip that occurs on a Möbius band.

Researchers in mirror symmetry and algebraic geometry represent singular integral affine spheres via polyhedra in which the flat faces are integral affine and all nodes lie on the edges [HZ].

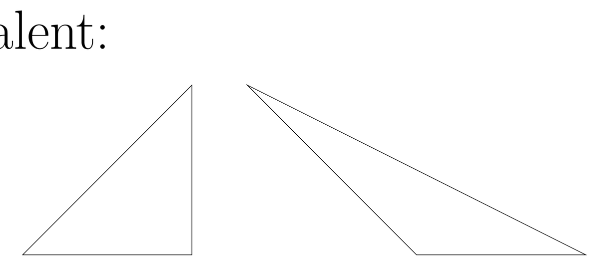
Our goals were to:

- "Unfold" polyhedral models of singular integral affine spheres.
- Give a step by step process for unfolding polyhedral models to get large integral affine coordinate charts.

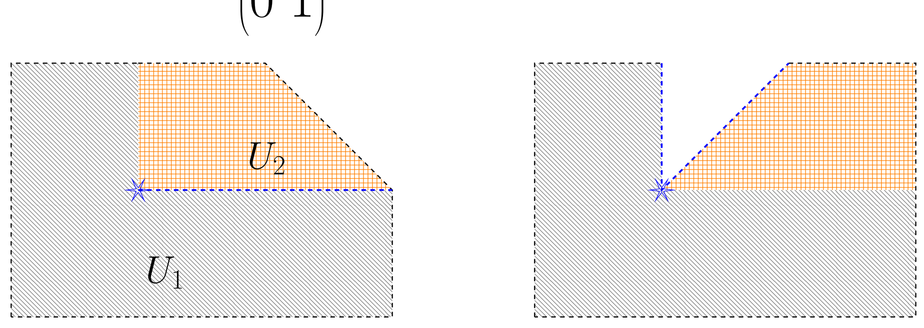
[HZ] Christian Haase and Ilia Zharkov. 'Integral affine structures on spheres and torus fibrations of Calabi-Yau toric hypersurfaces I.' arxiv.org/abs/math/0205321

Background

- The standard integral affine structure $\mathcal{A}_0$ on $\mathbb{R}^2$ is the geometry whose group of isomorphisms (congruences) is $GL(2,\mathbb{Z} \ltimes \mathbb{R}^2)$.
- A line segment with endpoints (x_0, y_0) and (x_1, y_1) in $(\mathbb{R}^2, \mathcal{A}_0)$ has integral affine length $\lambda \in \mathbb{R}^+$ if $(x_1, y_1) = (x_0 + m\lambda, y_0 + n\lambda)$ for some relatively prime $m, n \in \mathbb{Z}$.
- In $(\mathbb{R}^2, \mathbb{A}_0)$, the following triangles are integral affine equivalent:



- In a singular integral affine surface, a node has a neighborhood that, up to integral affine equivalence, is covered by two coordinate charts with transition maps given by $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ on U_1 and by $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ on U_2. Together they encode a clockwise monodromy of $\begin{pmatrix}1&1\\0&1\end{pmatrix}$.



For a node at (x_0, y_0), there is some vector (a, b) such that its clockwise monodromy is

$$\begin{pmatrix}1-ab & a^2\\ -b^2 & 1+ab\end{pmatrix},$$

which leaves fixed the eigenline

$$b(x - x_0) = a(y - y_0).$$

Warm up: Unfolding a Cylinder

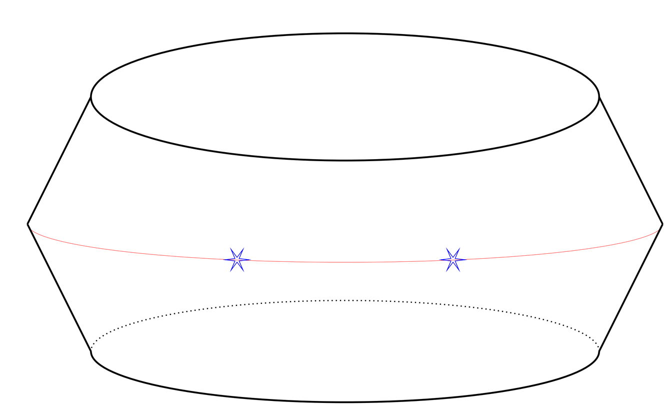
To unfold a cylinder with nodes:

- Cut from one edge to the other, avoiding nodes. (Green)
- Cut from each node to the boundary, with no cuts crossing each other. (Blue)

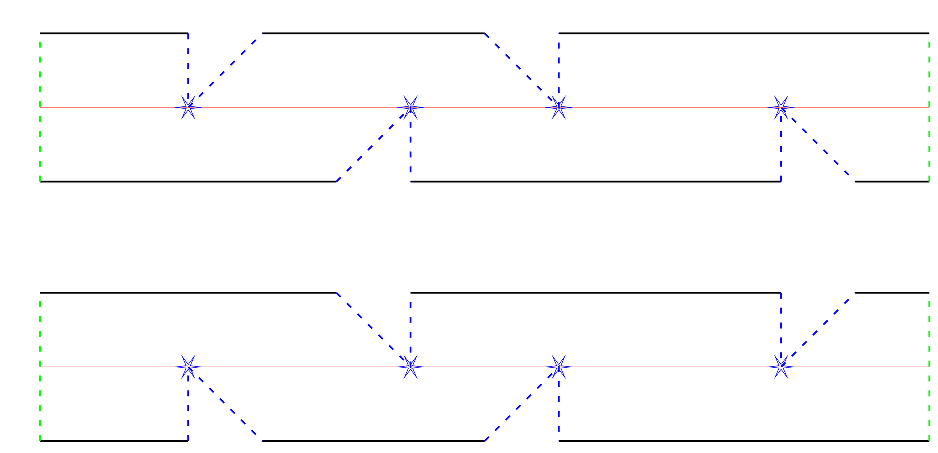
The complement of the cuts is isomorphic to a subset of $(\mathbb{R}^2, \mathcal{A}_0)$ that gives a big integral affine coordinate chart. Two coordinate charts are needed to cover the entire cylinder.

Diagram conventions: Red lines are eigenlines; dashed lines indicate cuts.

Cylinder with 4 nodes



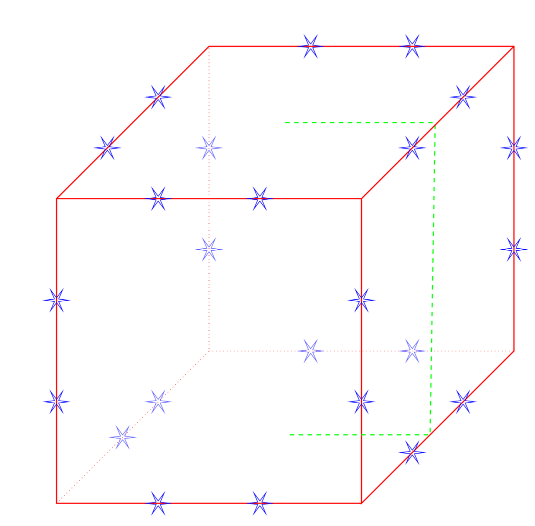
Coordinate charts



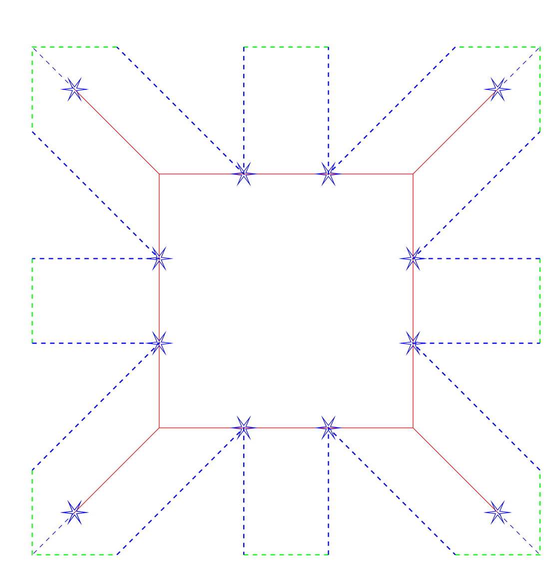
Unfolding a Cube

A cube with two nodes on each edge is a polyhedral model of a singular integral affine sphere. We unfolded such a cube model in two ways to get three integral affine coordinate charts that cover the entire sphere (minus the nodes). With respect to the standard basis on $(\mathbb{R}^2, \mathcal{A}_0)$, the clockwise monodromy matrices for the nodes in these coordinate charts are:

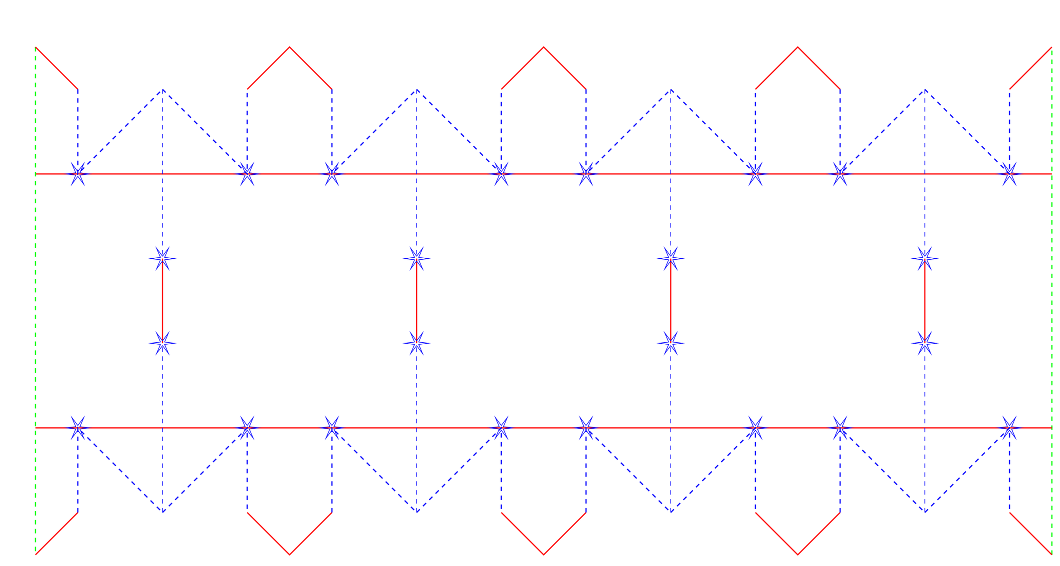
$$\begin{pmatrix}1&0\\-1&1\end{pmatrix}, \begin{pmatrix}0&1\\-1&2\end{pmatrix}, \begin{pmatrix}2&1\\-1&0\end{pmatrix} \text{ and } \begin{pmatrix}1&1\\0&1\end{pmatrix}.$$



Cube with 24 nodes



Half Sphere Minus Cuts

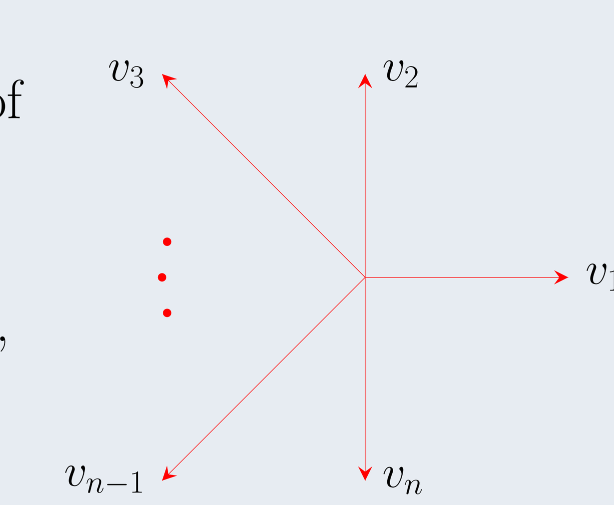


Whole Sphere Minus Cuts

Guidelines for Unfolding

- Start at a face whose boundary is made up of eigenlines, or at a vertex of the polyhedron.
- Make outward cuts from nodes so that cuts do not cross each other.
- At a vertex where eigenlines meet, require $|v_1 v_2| = |v_2 v_3| = \cdots = |v_n v_1|$, also known as a Delzant condition.

Note: An "opening" slice (not from a node, and indicated here in green) must also be made. Where to slice sometimes becomes apparent as one tries to unfold.



Delzant Condition