Extremal Numbrix Puzzles

Faith Hensley & Ashley Peper   Advisor: Dr. Lauren Keough

Overview

Numbrix is a puzzle in Parade magazine in which players are presented with a 9×9 grid with certain cells filled in with integers between 1 and 81. These predetermined integers in certain cells are the puzzle's clues. In our research, we generalize Numbrix puzzles to be played on any $m \times n$ grid, where $m, n \in \mathbb{N}$, and focus on the minimum number of clues required to guarantee that there exists a unique solution given those clues.

How to Play Numbrix

Numbrix is played on an $m \times n$ grid where the numbers 1 through mn must be filled in such that they create a path through the grid. Consecutive numbers must go in adjacent cells, and every number 1 through mn must be used exactly once.

1?	**4**	
2	1?	**6**
1?	**8**	

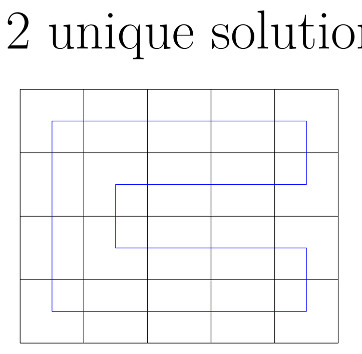
3	**4**	5
2	9	**6**
1	**8**	7

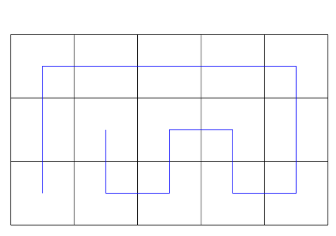
Prior Research

A set of clues **defines** a puzzle if there exists a unique solution provided such clues.

Theorem (Hanson & Nash): *One clue does not define an $m \times n$ puzzle if m or n is even.*

Any puzzle with m or n even can have a Hamiltonian circuit drawn through it. Then one can fill in a solution in either direction along the Hamiltonian circuit, providing 2 unique solutions.





Theorem (Hanson & Nash): *For an $m \times n$ grid with $3 \leq m \leq n$, the minimum number of clues required to define a Numbrix puzzle is less than or equal to $\lceil \frac{m}{2} \rceil$.*

5	4	3	2	1
6	7	8	9	10
15	14	13	12	11
16	17	18	19	20
25	24	23	22	21

A 5×5 Puzzle Defined by 3 Clues

Our Research Question

Can fewer than $\lceil \frac{m}{2} \rceil$ clues define an $m \times n$ Numbrix puzzle?

References

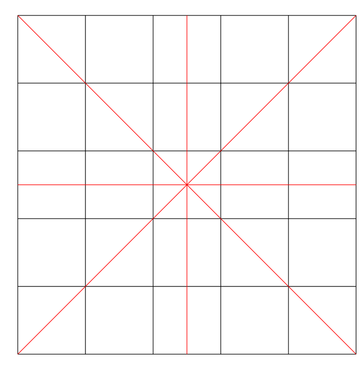
[1] Hanson, Mary Grace and Nash, David A. *Minimal and maximal Numbrix puzzles.* Pi Mu Epsilon Journal, Vol. 14, 2018.

[2] Thompson, Gerald L. *Hamiltonian Tours and Paths in Rectangular Lattice Graphs.* Mathematics Magazine, Vol. 50, 1977.

Can 1 Clue Define a Puzzle?

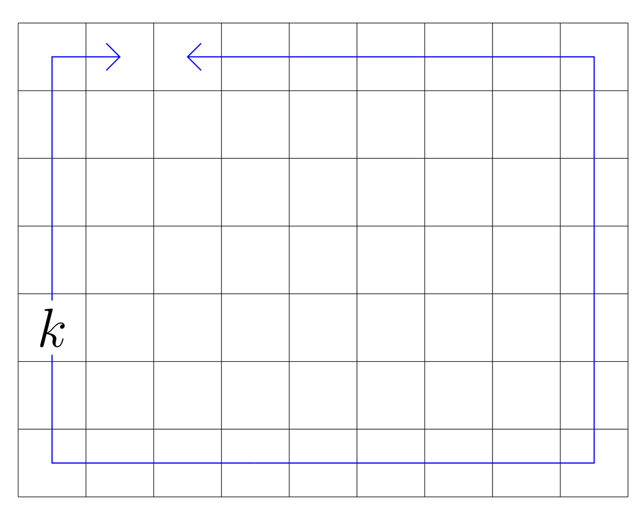
We begin by considering puzzles with a single clue. Since Hanson and Nash have already proven that one clue cannot define puzzles with an even dimension, we only need to examine odd $\times$ odd puzzles.

- **Clues on Lines of Symmetry**
  Given any clue on a line of symmetry, if there exists a solution, another solution can be found by reflecting the puzzle over the line of symmetry.

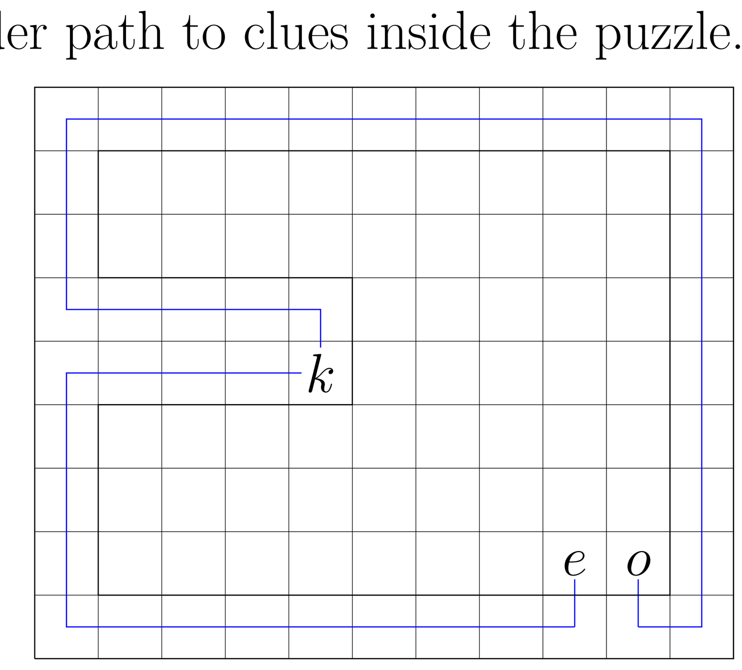


15	4	3	2	1
14	**5**	6	7	8
13	12	11	10	9

- **Border Clues**
  Given a clue on the border of the puzzle, we can draw a path around the border to be filled in either direction, providing 2 unique solutions.



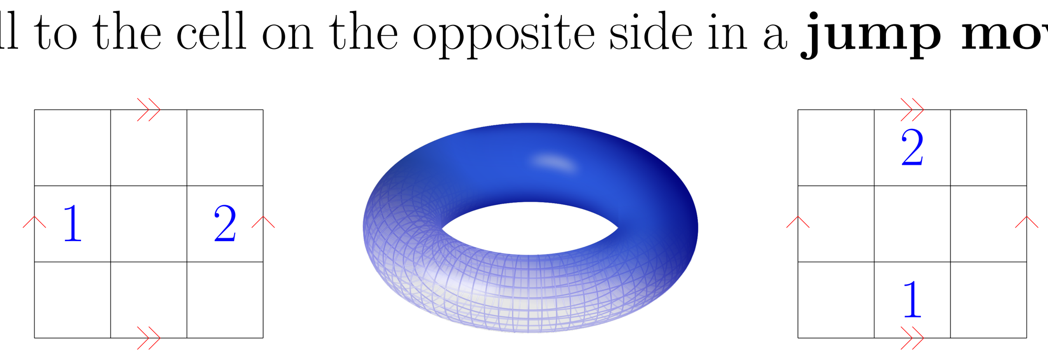
- **Other Clues**
  We can extend the above technique by connecting the border path to clues inside the puzzle.



Theorem (Hensley & Peper): *One clue cannot define an $m \times n$ puzzle with $m, n > 1$.*

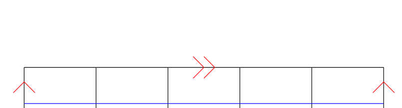
Numbrix on a Flat Torus

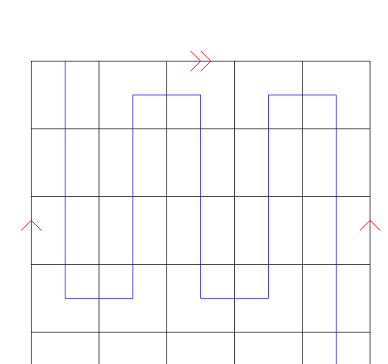
When playing on a torus we can move from a border cell to the cell on the opposite side in a **jump move**.



Can 1 Clue Define a Torus Puzzle?

Every $1 \times n$ torus puzzle has a Hamiltonian circuit. In fact, a Hamiltonian circuit can be drawn in any Numbrix puzzle on a torus.





Theorem (Hensley & Peper): *One clue cannot define a Numbrix puzzle on a torus larger than 1×2 or 2×1.*

Diagonal Numbrix Puzzles

Diagonal Numbrix puzzles are puzzles which allow for all normal Numbrix moves as well as **diagonal moves** in any of the four diagonal directions.

1	**2**
4	3

1	**2**
3	4

Puzzles defined under normal Numbrix rules may not be defined as diagonal Numbrix puzzles.

Checkerboard Arrangement

Theorem (Hensley & Peper): *The minimum number of clues needed to define a diagonal Numbrix puzzle is at most $\lfloor \frac{mn}{2} \rfloor$.*

13	**14**	15	**16**	17	**18**
12	11	**10**	21	**20**	19
7	**8**	9	**22**	23	**24**
6	5	**4**	27	**26**	25
1	**2**	3	**28**	29	**30**

This arrangement is **minimal**, but not **minimum**!

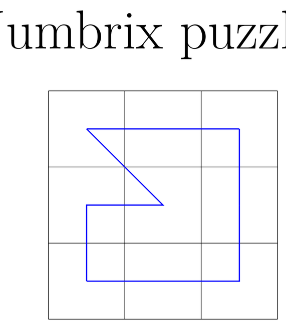
Upper Bound for the Minimum

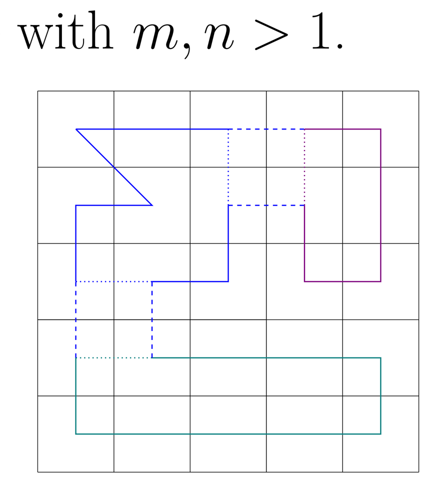
Theorem (Hensley & Peper): *The minimum number of clues needed to define a diagonal Numbrix puzzle is at most $\lfloor \frac{n+1}{3} \rfloor (m+1)$.*

1	**11**	**12**	13	**23**	**24**	25
2	**9**	10	14	**21**	22	26
3	**7**	8	15	**19**	20	27
4	**5**	6	16	**17**	18	28

Can 1 Clue Define a Diagonal Puzzle?

A Hamiltonian circuit can be drawn through any diagonal Numbrix puzzle with $m, n > 1$.





Theorem (Hensley & Peper): *One clue cannot define an $m \times n$ diagonal Numbrix puzzle with $m, n > 1$.*

Further Research

- Prove that 2 clues do not define an $m \times n$ Numbrix puzzle with $m, n \geq 5$.
- Explore minimum number of clues to define a Numbrix puzzle on a torus.

Acknowledgements

This work was partially supported by National Security Agency grant H98230-19-1-0015, which funds a Research Experiences for Undergraduates program at Grand Valley State University. We'd like to thank the NSA, GVSU, and our mentor Lauren Keough.