

# The Tambara Structure of the Trace Ideal

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Ring Theory  $\leftrightarrow$  Tambara Functor Theory

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Trace homomorphism:

$$A \rightarrow GW$$

Dress map:

$$\underline{A} \rightarrow \underline{GW}$$

Kernel

Trace ideal  $\mathcal{TI}$

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Kernel

Trace ideal  $\mathcal{TI}$

Goal: Determine  $\mathcal{TI}$

Then

$$\underline{GW} \cong \underline{A} / \mathcal{TI}$$

when the Dress map is surjective.

# The Basic Ingredients

## Our Main Example

Cyclic group with  $N$  elements:  $C_N$ .

Finite field with  $q$  elements:  $\mathbb{F}_q$  (for  $q$  a power of an odd prime).

$\mathbb{F}_q \subseteq \mathbb{F}_p \iff q \mid p$ , i.e.  $p = q^N$ . Then  $\text{Gal}(\mathbb{F}_p/\mathbb{F}_q) = C_N$ .

## The Less Basic Ingredients

### Tambara functors (D. Tambara [4], 1993)

Specified by data:

- A commutative ring for each subgroup of  $G$
- Tambara structure maps **restriction**, **transfer**, **norm**, and **conjugation** satisfying various compatibility conditions and commutative diagrams

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Examples we care about:

- **Burnside** Tambara functor  $\underline{A}$
- **Grothendieck-Witt** (Galois) Tambara functor  $\underline{GW}$

# The Burnside functor $\underline{A}$ on $C_p$

$$n[C_p/e] + m[C_p/C_p] := nt_p + m \text{ and } t_p^2 = pt_p$$

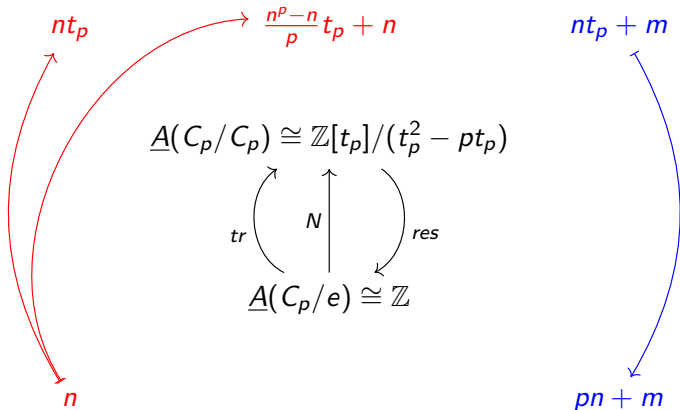
$$\underline{A}(C_p/C_p) \cong \mathbb{Z}[t_p]/(t_p^2 - pt_p)$$

$$\begin{array}{ccc} & \uparrow N & \\ \text{tr} \curvearrowright & & \curvearrowleft \text{res} \\ \underline{A}(C_p/e) & \cong & \mathbb{Z} \end{array}$$

$$n[e/e] := n$$



# The Burnside functor $\underline{A}$ on $C_p$



# The Grothendieck-Witt functor $\underline{GW}$ on $\mathbb{F}_q \subseteq \mathbb{F}_{q^p}$

$$n\langle 1 \rangle \oplus \langle \alpha \rangle$$

$$\begin{array}{ccc}
 & \underline{GW}(\mathbb{F}_q) & \\
 \text{tr} \curvearrowright & \uparrow N & \curvearrowleft \text{res} \\
 & \underline{GW}(\mathbb{F}_{q^p}) & 
 \end{array}$$

$$n\langle 1 \rangle \oplus \langle \beta \rangle$$

# The Grothendieck-Witt functor $\underline{GW}$ on $\mathbb{F}_q \subseteq \mathbb{F}_{q^p}$

$$\begin{array}{ccc}
 & \underline{GW}(\mathbb{F}_q) & \\
 \text{tr} \curvearrowright & \uparrow N & \curvearrowleft \text{res} \\
 & \underline{GW}(\mathbb{F}_{q^p}) &
 \end{array}$$

restriction:  $\langle 1 \rangle \mapsto \langle 1 \rangle$

$\langle \alpha \rangle \mapsto \langle \beta \rangle$

transfer:  $\langle 1 \rangle \mapsto p\langle 1 \rangle$

$\langle \beta \rangle \mapsto (p-1)\langle 1 \rangle \oplus \langle \alpha \rangle$

norm:  $n\langle 1 \rangle \mapsto n^p\langle 1 \rangle$

$(n-1)\langle 1 \rangle \oplus \langle \beta \rangle \mapsto (n^p-1)\langle 1 \rangle \oplus \langle \alpha^n \rangle$

# The Dress Map

## Definition

- For rings, trace homomorphism (A. Dress [2], 1971)
- For Tambara functors, **Dress map**  $\mathcal{D}$  is given by trace homomorphism at each level

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$$t_p \mapsto p\langle 1 \rangle$$

$$1 \mapsto \langle 1 \rangle$$

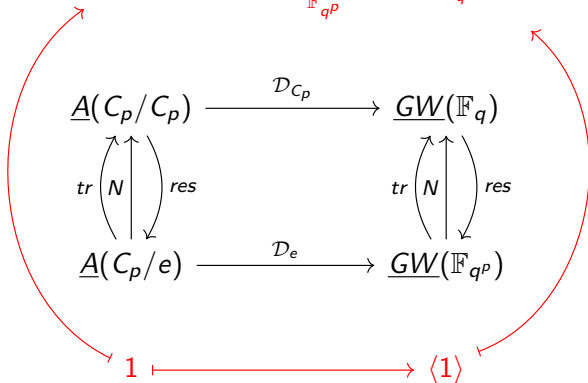
$$\begin{array}{ccc}
 \underline{A}(C_p/C_p) & \xrightarrow{\mathcal{D}_{C_p}} & \underline{GW}(\mathbb{F}_q) \\
 \left. \begin{array}{c} \uparrow \\ \text{tr} \left( \begin{array}{c} \uparrow \\ N \\ \downarrow \end{array} \right) \\ \downarrow \\ \text{res} \end{array} \right\} & & \left. \begin{array}{c} \uparrow \\ \text{tr} \left( \begin{array}{c} \uparrow \\ N \\ \downarrow \end{array} \right) \\ \downarrow \\ \text{res} \end{array} \right\} \\
 \underline{A}(C_p/e) & \xrightarrow{\mathcal{D}_e} & \underline{GW}(\mathbb{F}_{q^p}) \\
 1 & \mapsto & \langle 1 \rangle
 \end{array}$$

## Example: The Whole Picture

$$\begin{array}{ccc}
 \underline{A}(C_p/C_p) & \xrightarrow{\mathcal{D}_{C_p}} & \underline{GW}(\mathbb{F}_q) \\
 \left. \begin{array}{c} \uparrow \\ \text{tr} \left( \begin{array}{c} \uparrow \\ N \\ \downarrow \end{array} \right) \\ \downarrow \end{array} \right\} \text{res} & & \left. \begin{array}{c} \uparrow \\ \text{tr} \left( \begin{array}{c} \uparrow \\ N \\ \downarrow \end{array} \right) \\ \downarrow \end{array} \right\} \text{res} \\
 \underline{A}(C_p/e) & \xrightarrow{\mathcal{D}_e} & \underline{GW}(\mathbb{F}_{q^p})
 \end{array}$$

## Example: The Whole Picture

$$\mathrm{tr}_{e^{C_p}}^{C_p}(1) = t_p \longmapsto \mathrm{tr}_{\mathbb{F}_{q^p}}^{\mathbb{F}_q^e} \langle 1 \rangle = \mathrm{tr}_{\mathbb{F}_q}^{\mathbb{F}_{q^p}} \langle 1 \rangle = \rho \langle 1 \rangle$$



# The Trace Ideal

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The **trace ideal** is the kernel of the Dress map,

$$\mathcal{TI} = \{\ker(\mathcal{D}_H)\}_{H \leq G}$$



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## Goal

Determine trace ideal (as Tambara ideal), find generators.

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## Definition

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## Goal

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## Theorem

For cyclic groups, there is *one* generator!

## Other Results and Future Work

### 1. Arbitrary cyclic extensions (non-finite fields)

- For  $\mathbb{C}/\mathbb{R}$ , e.g., the trace ideal is 0, implying  $\underline{GW} \cong \underline{A}$

### 2. Profinite extensions of finite fields

- Quadratic closure  $\mathbb{F}_{q^{2^\infty}}$  and the algebraic closure  $\overline{\mathbb{F}_q}$

### 3. Prime ideals of $\underline{A}$

- In progress...

# References



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Thank You!  
(Questions?)