Symbolic Powers of Defining Ideals of Veronese Rings

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Overview

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Ideals and Varieties

Variety

Example

Let $f \in R = \mathbb{R}[x, y, z]$ defined by $f(x, y, z) = 2x - y + 3z$.
Then $V(f) := \{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = 0\}$ is a plane.

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Definition

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring with $\mathbb{C}$- coefficients and variables $x_1, \ldots, x_n$. A variety is the set of common zeroes in $\mathbb{C}^n$ of a collection of polynomials $f_i \in R$. The variety associated to the set $\{f_1, \ldots, f_m\}$ is written $V(f_1, \ldots, f_m)$.

If p is a point in the variety, then p is also a zero of any polynomial combination

$$\sum_{i=1}^{m} g_i(x_1, \ldots, x_n) f_i(x_1, \ldots, x_n)$$

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Observation

The variety $V(f_1, \ldots, f_m)$ depends only on the ideal I generated by $\{f_1, \ldots, f_m\}$.

Ideals and Varieties

Given a variety of an ideal J, we can form the set $I(V(J))$ of all polynomials vanishing on this variety. It is easy to check the set is an ideal.

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If $J \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is radical, then $J = I(V(J))$.

Definition

Given an ideal I in a ring R, the radical of I is $\sqrt{I} = \{f \in R : f^n \in I \text{ for some } n \in \mathbb{N}\}$.
An ideal I is radical if $\sqrt{I} = I$.

Primary Decomposition

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If I, J in a polynomial ring R, then $V(I \cap J) = V(I) \cup V(J)$.

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Example

Consider the ideal $I = (xz, yz) = (z) \cap (x, y)$ in $\mathbb{R}[x, y, z]$.
In $\mathbb{R}^3$, $V(z)$ corresponds to the xy-plane and $V(x, y)$ corresponds to the z-axis, then $V(I) = V(z) \cup V(x, y)$.

Primary Decomposition

Given an ideal, we would like to decompose it as an intersection of simpler ideals.

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Example

In the ring of integers $\mathbb{Z}$, suppose a positive integer n has prime factorization $n = p_1^{a_1} \dots p_r^{a_r}$, then the ideal $(n) = (p_1^{a_1}) \cap \dots \cap (p_r^{a_r})$.
For example, for $60 = 2^2 \cdot 3 \cdot 5$, we have $(60) = (4) \cap (3) \cap (5)$.

Primary Decomposition

Definition

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A primary decomposition of an ideal I consists of primary ideals $Q_1, \ldots, Q_n$ such that $I = \cap_{i=1}^{n} Q_i$. A primary decomposition $I = \cap_{i=1}^{n} Q_i$ is irredundant if $\cap_{i \neq j} Q_i \neq I$ for each $j \in \{1, \ldots, n\}$ and $\sqrt{Q_i} \neq \sqrt{Q_j}$ for all $i \neq j$.

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Theorem

Any ideal in a Noetherian ring has a primary decomposition.

Symbolic Powers

Symbolic Powers

$\mathbb{C}[x]$

- f vanishes at p
- f vanishes to order $\geq k$ at p

- $(x-p) \mid f$ or $f \in (x-p)$
  $x = p$ is a root of f
- $(x-p)^k \mid f$ or $f \in (x-p)^k$
  $x = p$ is a root of $f, f', \ldots, f^{(k-1)}$

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$\mathbb{C}[x_1, \ldots, x_n]$

- f vanishes to order $\geq k$ at $(p_1, \ldots, p_n)$
- $f \in (x_1 - p_1, \ldots, x_n - p_n)^k$
  $x = (p_1, \ldots, p_n)$ is a root of
  $\frac{\partial^{d_1}}{\partial x_1^{d_1}} \cdots \frac{\partial^{d_n}}{\partial x_n^{d_n}} f$ for all $d_1 + \cdots + d_n < k$

Symbolic Powers

Definition (Zariski-Nagata)

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and I a radical ideal in R. Then the k-th symbolic power of I is
$I^{(k)} = \{f \in R | f \text{ vanishes to order} \geq k \text{ at every } x \in V(I)\}$.

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- $\mathbb{C}[x_1, \ldots, x_n]$
- radical ideal I
- symbolic power $I^{(k)}$
- $\mathbb{C}^n$
- vanish on variety $V(I)$
- vanish to order k over variety

Veronese Ring and Ideal

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$S_{2,3} := (\mathbb{C}[x_1, x_2])_3 = \mathbb{C}[x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3]$.

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Let $R = \mathbb{C}[t_1, t_2, t_3, t_4]$.

There is a surjective ring homomorphism $\pi : R \to S_{2,3}$ defined by $t_1 \mapsto x_1^3$, $t_2 \mapsto x_1^2x_2$, $t_3 \mapsto x_1x_2^2$, $t_4 \mapsto x_2^3$, and $I_{2,3} := \ker(\pi)$.

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Note that

$$\pi(t_1t_3 - t_2^2) = x_1^4x_2^2 - x_1^4x_2^2 = 0$$
$$\pi(t_1t_4 - t_2t_3) = x_1^3x_2^3 - x_1^3x_2^3 = 0$$
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$$\pi(t_2 t_4 - t_3^2) = x_1^2 x_2^4 - x_1^2 x_2^4 = 0$$

In fact, $I_{2,3} = \ker(\pi) = (t_1 t_3 - t_2^2, t_1 t_4 - t_2 t_3, t_2 t_4 - t_3^2)$, and
$I_{2,3} \longleftrightarrow V(t_1 t_3 - t_2^2, t_1 t_4 - t_2 t_3, t_2 t_4 - t_3^2) \subset \mathbb{C}^4$.

Veronese Ring and Ideal

Definition

Let $S_n = \mathbb{C}[x_1, \ldots, x_n]$. The d-th Veronese in n variables $S_{n,d} := (\mathbb{C}[x_1, \ldots, x_n])_d = \mathbb{C}[x_1^d, x_1^{d-1}x_2, \ldots, x_n^d]$. Let $R = k[t_1, t_2, \ldots, t_{\binom{n+d-1}{d}}]$. There is a surjective ring homomorphism

$$\begin{array}{ccc} R & \longrightarrow & S_{n,d} \\ t_1 & \longmapsto & x_1^d \\ t_2 & \longmapsto & x_1^{d-1}x_2 \\ & \vdots & \\ t_{\binom{n+d-1}{d}} & \longmapsto & x_n^d \end{array}$$

We write $I_{n,d}$ for the kernel of this map.

Our Results

The ideal $I_{n,d}$ $(n, d \geq 2)$ is generated by 2-minors of a $n \times \binom{n+d-2}{d-1}$ matrix.

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Example

$I_{2,3} = I_2 \begin{bmatrix} t_{(3,0)} & t_{(2,1)} & t_{(1,2)} \\ t_{(2,1)} & t_{(1,2)} & t_{(0,3)} \end{bmatrix}$ where $t_{(3,0)} \mapsto x_1^3,\ t_{(2,1)} \mapsto x_1^2 x_2,$ $t_{(1,2)} \mapsto x_1 x_2^2,\ t_{(0,3)} \mapsto x_2^3.$

Example

$$I_{2,4} = I_2 \begin{bmatrix} t_{(4,0)} & t_{(3,1)} & t_{(2,2)} & t_{(1,3)} \\ t_{(3,1)} & t_{(2,2)} & t_{(1,3)} & t_{(0,4)} \end{bmatrix}.$$

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Example

$$I_{3,3} = I_2 \begin{bmatrix} t_{(3,0,0)} & t_{(2,1,0)} & t_{(2,0,1)} & t_{(1,2,0)} & t_{(1,1,1)} & t_{(1,0,2)} \\ t_{(2,1,0)} & t_{(1,2,0)} & t_{(1,1,1)} & t_{(0,3,0)} & t_{(0,2,1)} & t_{(0,1,2)} \\ t_{(2,0,1)} & t_{(1,1,1)} & t_{(1,0,2)} & t_{(0,2,1)} & t_{(0,1,2)} & t_{(0,0,3)} \end{bmatrix}.$$

Macaulay2 Computations

We know $I_{n,d}^b \subseteq I_{n,d}^{(b)}$ for all $b \in \mathbb{N}$.

We use Macaulay2 to find the smallest a such that $I_{n,d}^{(a)} \subseteq I_{n,d}^b$ for some n, d, b.

n	d	b	a	n	d	b	a
2	4	2	3	3	2	2	3
		3	4			3	4
		4	5			4	5
		5	7			5	7
		6	8			6	8
		7	9			7	9
2	5	2	3			8	11
		3	4	3	3	2	3
		4	5			3	4
2	6	2	3	4	2	2	3
		3	4			3	4
2	7	2	3	5	2	2	3
		3	4				
2	8	2	3				

$I_{2,d}^{(2b-1)} \subseteq I_{2,d}^{b}$ for all $b \in \mathbb{N}$.

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$in_{<}(I_{n,d}^{(2b-1)}) \subseteq in_{<}(I_{n,d}^{b})$ for all $b \in \mathbb{N}$.

- For $d = 1, 2, 3$, $I^a_{2,d} = I^{(a)}_{2,d}$.
- For $d \geq 4$ and $a \geq 2$, an irredundant primary decomposition for $I^a_{2,d}$ is

$$I^a_{2,d} = I^{(a)}_{2,d} \cap \mathfrak{m}^{2a}$$

  where $\mathfrak{m} = \left(t_{(d,0)}, t_{(d-1,1)}, t_{(d-2,2)}, \ldots, t_{(1,d-1)}, t_{(0,d)}\right)$ is the ideal of all the t variables.

For $d \geq 2$, the least degree of an element in $I_{2,d}^{(a)}$ is

- $\lceil \frac{(d+2)a}{d} \rceil$ if d is even;
- $\lceil \frac{(d+1)a}{d-1} \rceil$ if d is odd.

Example

For $d \geq 4$, the minimal degree of an element in the $I_{2,d}^{(2)}$ is 3.

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For $d \geq 4$, the minimal degree of an element in the $I_{2,d}^{(2)}$ is 3.
In particular, for $d = 4$,

$$[I_{2,4}^{(2)}]_3 \subseteq I_3 \begin{bmatrix} t_{(4,0)} & t_{(3,1)} & t_{(2,2)} \\ t_{(3,1)} & t_{(2,2)} & t_{(1,3)} \\ t_{(2,2)} & t_{(1,3)} & t_{(0,4)} \end{bmatrix}$$

Example

$$I_{2,4} = I_2 \begin{bmatrix} t_{(4,0)} & t_{(3,1)} & t_{(2,2)} & t_{(1,3)} \\ t_{(3,1)} & t_{(2,2)} & t_{(1,3)} & t_{(0,4)} \end{bmatrix} = I_2 \begin{bmatrix} t_1 & t_2 & t_3 & t_4 \\ t_2 & t_3 & t_4 & t_5 \end{bmatrix}.$$

Let $f = \det \begin{bmatrix} t_1 & t_2 & t_3 \\ t_2 & t_3 & t_4 \\ t_3 & t_4 & t_5 \end{bmatrix}$.

1. $I_{2,4}^{(2)} = I_{2,4}^2 + I_3 \begin{bmatrix} t_1 & t_2 & t_3 \\ t_2 & t_3 & t_4 \\ t_3 & t_4 & t_5 \end{bmatrix} = I_{2,4}^2 + (f)$.
2. $I_{2,4}^{(2k+1)} = I_{2,4}^{(2k)} I_{2,4}$ for all $k \geqslant 1$.
3. $I_{2,4}^{(2k+2)} = (I_{2,4}^{(2)})^{k+1}$ for all $k \geqslant 1$.

Thank You!