Finite Quotients of Braid Groups

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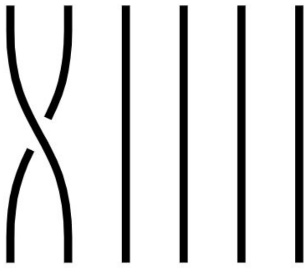
Joint Work with Alice Chudnovsky, Caleb Partin, and Kevin Kordek

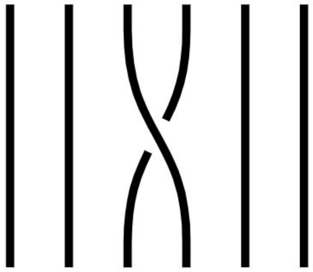
Consider…

Given two groups G, H, what are all the homomorphisms between them?

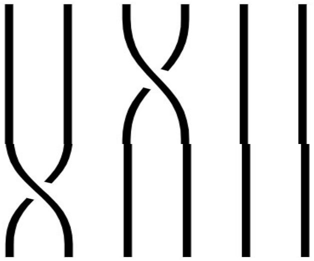
Braid Group



σ_1

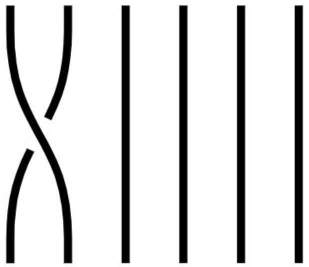


σ_3



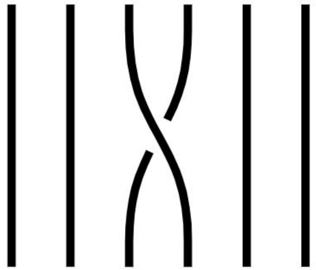
$\sigma_3\sigma_1$

Braid Group



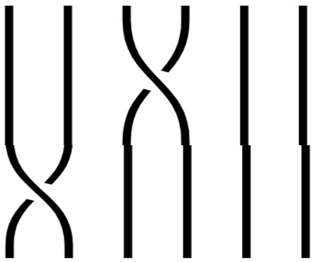
σ_1

(12)



σ_3

(34)



$\sigma_3\sigma_1$

$(12)(34)$

$$B_n \to S_n$$

Main Result

Finite Quotients of Braid Groups

Let G be a finite group and let $n \geq 5$.

If $B_n \to G$ is not a cyclic homomorphism, then $|G| \geq 2^{\lfloor \frac{n}{2} \rfloor - 1}(\lfloor \frac{n}{2} \rfloor)!$

Definition. Totally Symmetric Set (Kordek, Margalit)
A totally symmetric set X is a subset of a group G which satisfies two properties:

- All elements of the subset commute
- Any permutation in X can be achieved by conjugation in G.

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$$(1\ 2) \qquad (3\ 4) \qquad (5\ 6)$$

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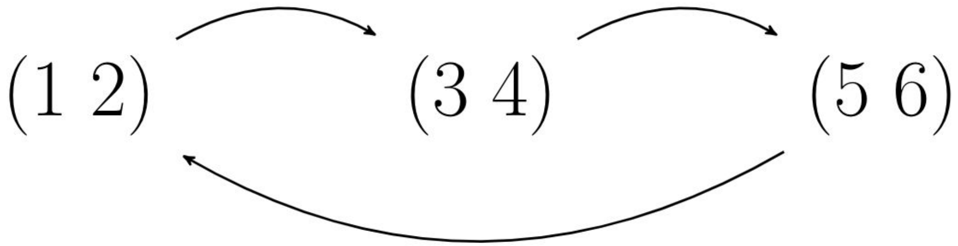

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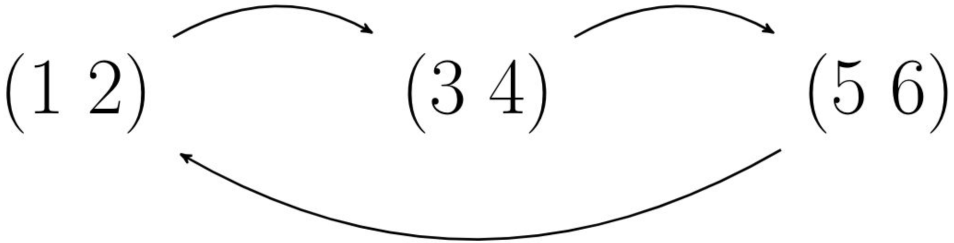
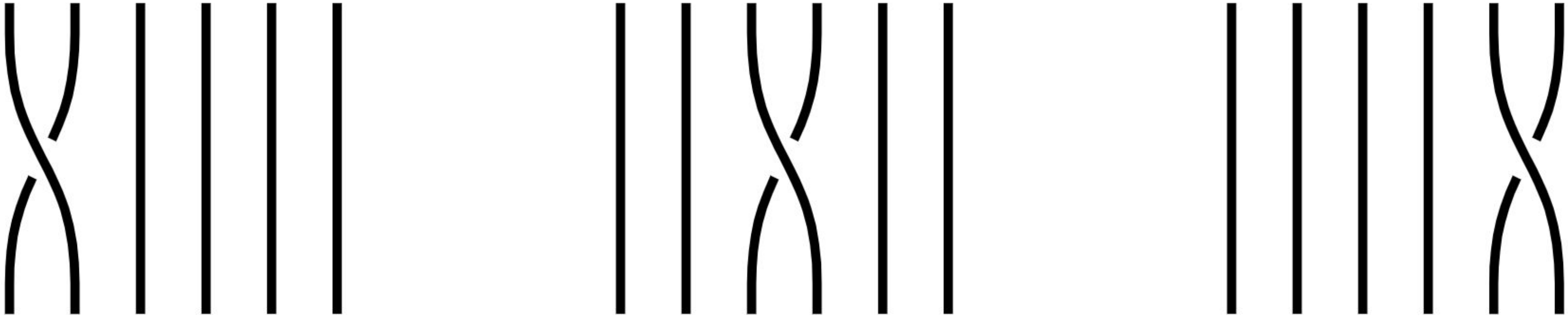


$$(135)(246)$$



σ_1 σ_3 σ_5

Fundamental Lemma of Totally Symmetric Sets

The image of a totally symmetric set under a homomorphism is a totally symmetric set.

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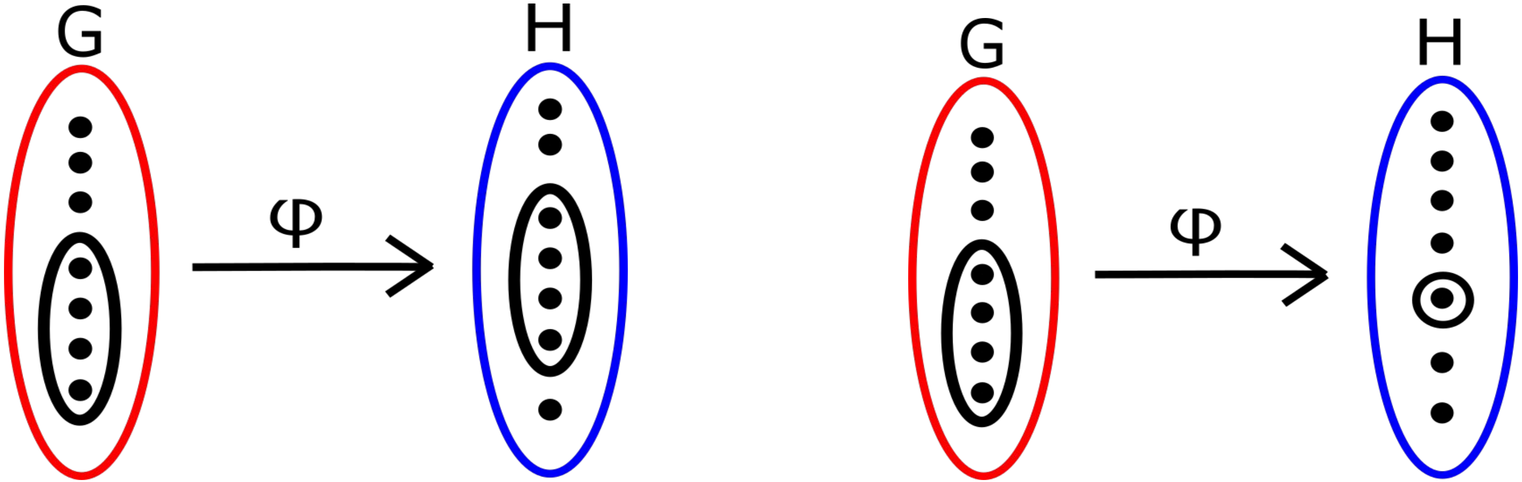
BUT IT GETS BETTER

Fundamental Lemma of Totally Symmetric Sets

The image of a totally symmetric set of size n under a homomorphism is a totally symmetric set of size n or 1.



$$B_n \to G$$

$$B_n \to G$$

$$\cup$$

$$S$$

Our favorite totally symmetric set

$$
\begin{array}{ccc}
B_n & \to & G \\
\cup & & \\
S & \to & \bar{S}
\end{array}
$$

Our favorite totally symmetric set

Image of S: also totally symmetric set

$$\begin{array}{ccc} B_n & \to & G \\ \cup & & \cup \\ S & \to & \bar{S} \end{array}$$

Our favorite totally symmetric set

Image of S: also totally symmetric set

The image of S is either…

singleton

Then the map is
cyclic

$$\langle\langle \sigma_1 \sigma_3^{-1} \rangle\rangle \cong B_n'$$

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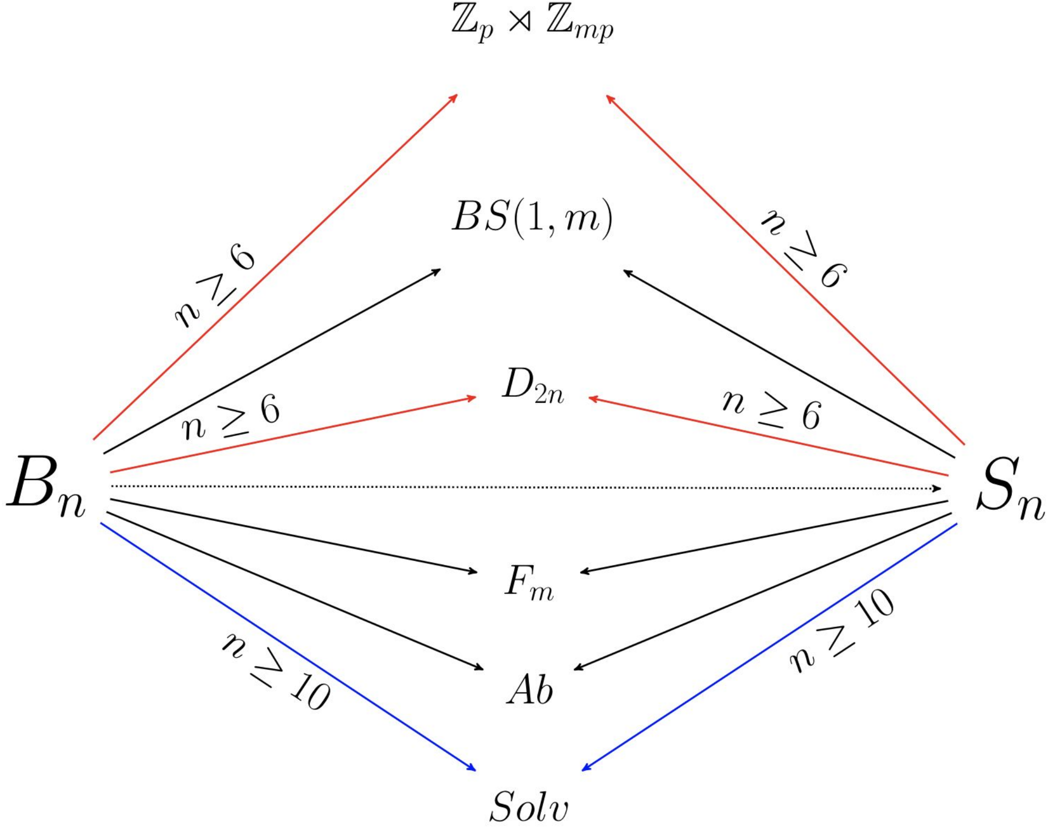

The image of S is either…

<u>a singleton</u>

Then the map is cyclic

<u>not a singleton</u>

The stabilizer of S in G
Its image is large: contains a copy of S_n

Case 2: Not singleton

$$B_n \to G$$

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$$\begin{array}{ccc} B_n \to G & & \\ & \cup & \\ & \Gamma_S \to S_{\lfloor \frac{n}{2} \rfloor} & \end{array}$$

Case 2: Not singleton

$$B_n \to G$$

$$\cup$$

$$\Gamma_S \to S_{\lfloor \frac{n}{2} \rfloor}$$

Stabilizer of the totally symmetric set

Case 2: Not singleton

$$B_n \to G$$

$$\cup$$

$$\Gamma_S \to S_{\lfloor \frac{n}{2} \rfloor}$$

Size of the totally symmetric set

Stabilizer of the totally symmetric set

Case 2: Not singleton

$$B_n \to G$$

$$\cup$$

$$1 \to K \to \Gamma_S \to S_{\lfloor \frac{n}{2} \rfloor} \to 1$$

Elements inducing trivial permutation

All the elements that permute S

Case 2: Not singleton

$$B_n \to G$$

$$\cup$$

$$1 \to K \to \Gamma_S \to S_{\lfloor \frac{n}{2} \rfloor} \to 1$$

$$|G| \geq |\Gamma_S| = |K| * |S_{\lfloor \frac{n}{2} \rfloor}|$$

Bounds on Sizes of Totally Symmetric Sets

G	$S(G)$
F_n	1
D_{2n}	2
$\mathbb{Z}/np \rtimes \mathbb{Z}/p$	2
$BS(1,n)$	1 *or* 2
$SL_2(\mathbb{C})$	2

G	$S(G)$
B_n	$\lfloor \frac{n}{2} \rfloor$
S_n	$\geq \lfloor \frac{n}{2} \rfloor$
$Aut(F_n)$	$\geq n$

G	$S(G)$
$G \times H$	$\max(S(G), S(H))$
Ab	1
Odd	1
$Solv$	≤ 4