

Finite Quotients of Braid Groups

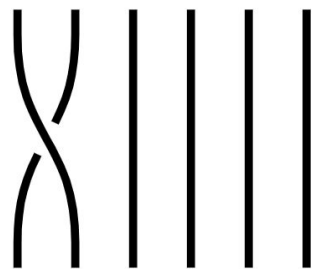
Lily Li

Joint Work with Alice Chudnovsky, Caleb Partin, and Kevin Kordek

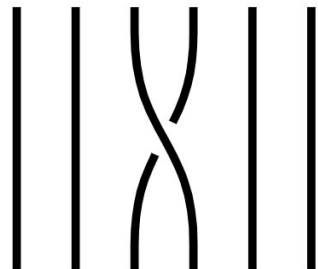
Consider...

*Given two groups G, H , what are all the homomorphisms
between them?*

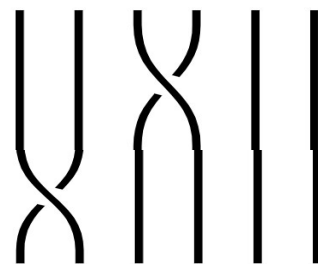
Braid Group



σ_1

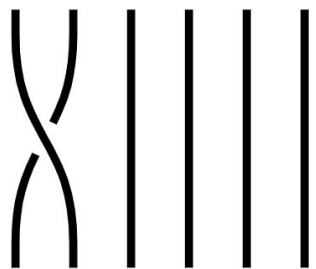
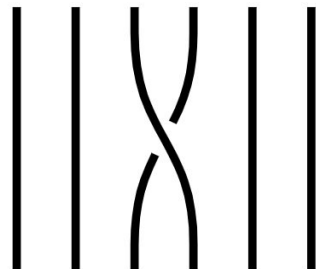
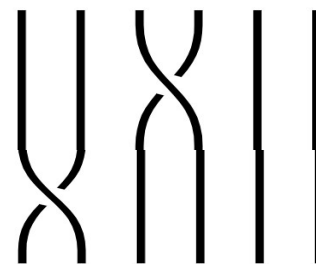


σ_3



$\sigma_3 \sigma_1$

Braid Group

 σ_1 (12)  σ_3 (34)  $\sigma_3\sigma_1$ $(12)(34)$

$$B_n \rightarrow S_n$$

Main Result

Finite Quotients of Braid Groups

Let G be a finite group and let $n \geq 5$.

If $B_n \rightarrow G$ is not a cyclic homomorphism, then $|G| \geq 2^{\lfloor \frac{n}{2} \rfloor - 1} (\lfloor \frac{n}{2} \rfloor)!$

Definition. Totally Symmetric Set (Kordek, Margalit)

A totally symmetric set X is a subset of a group G which satisfies two properties:

- All elements of the subset commute
- Any permutation in X can be achieved by conjugation in G .

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$$(1\ 2)$$

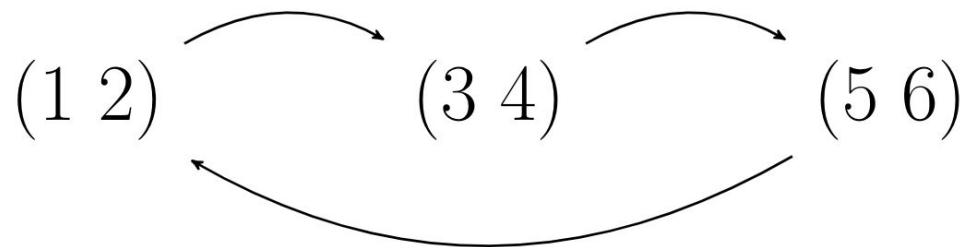
$$(3\ 4)$$

$$(5\ 6)$$

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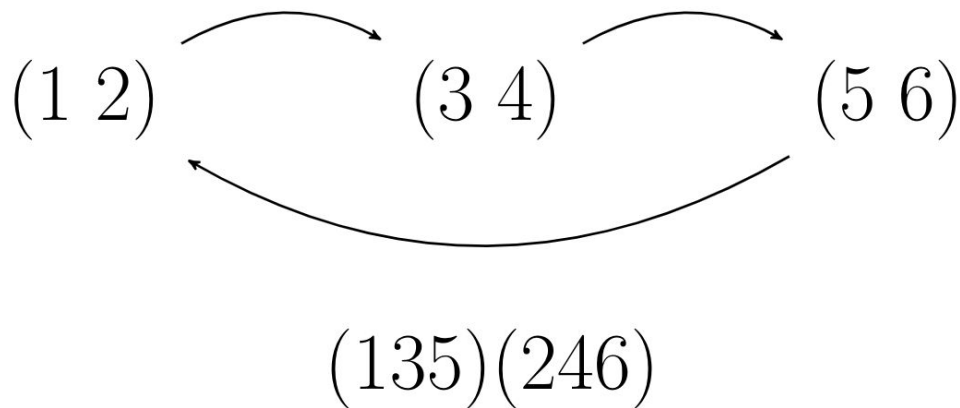
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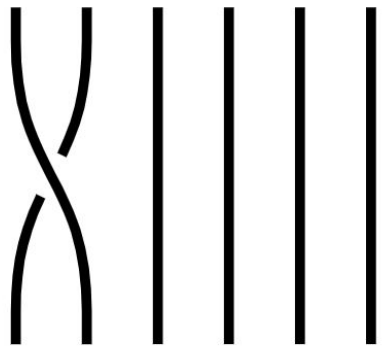


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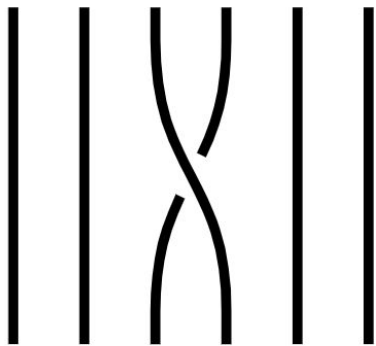
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σ_1



σ_3



σ_5

Fundamental Lemma of Totally Symmetric Sets

The image of a totally symmetric set under a homomorphism is a totally symmetric set.

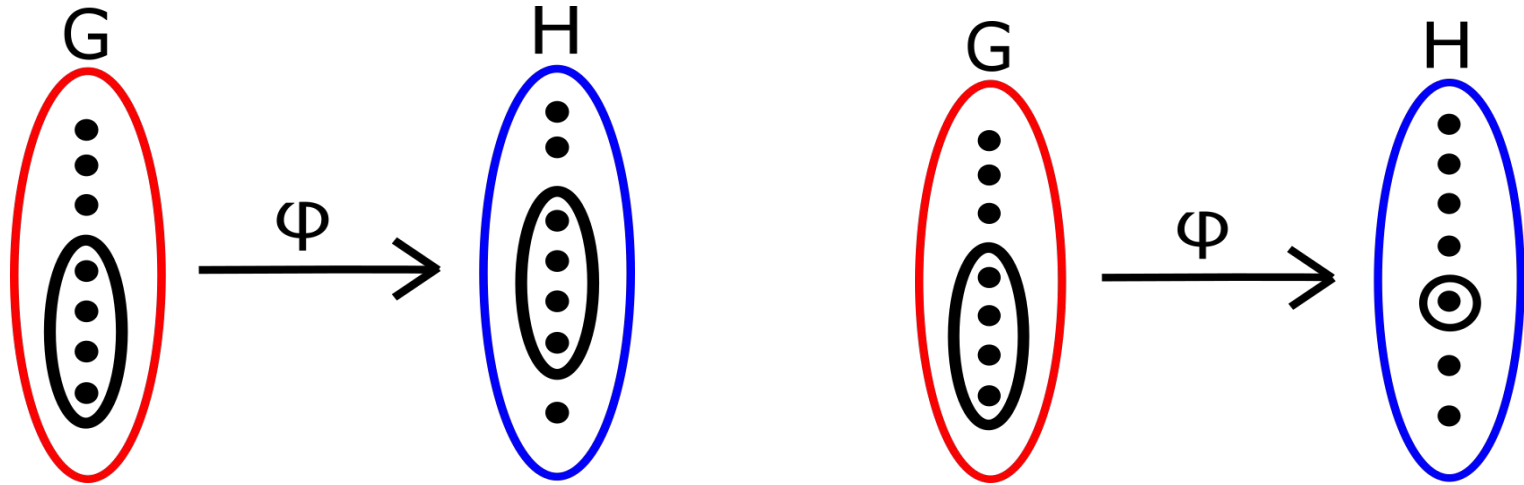
Fundamental Lemma of Totally Symmetric Sets

The image of a totally symmetric set under a homomorphism is a totally symmetric set.

BUT IT GETS BETTER

Fundamental Lemma of Totally Symmetric Sets

The image of a totally symmetric set of size n under a homomorphism is a totally symmetric set of size n or 1 .



$$B_n \rightarrow G$$

$$B_n \rightarrow G$$

\cup

S

Our favorite totally
symmetric set



$$B_n \rightarrow G$$

$$\cup$$

$$S \rightarrow \bar{S}$$

Our favorite totally
symmetric set

Image of S: also totally
symmetric set

$$B_n \rightarrow G$$

$$\cup \quad \cup$$

$$S \rightarrow \bar{S}$$

Our favorite totally
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Image of S: also totally
symmetric set

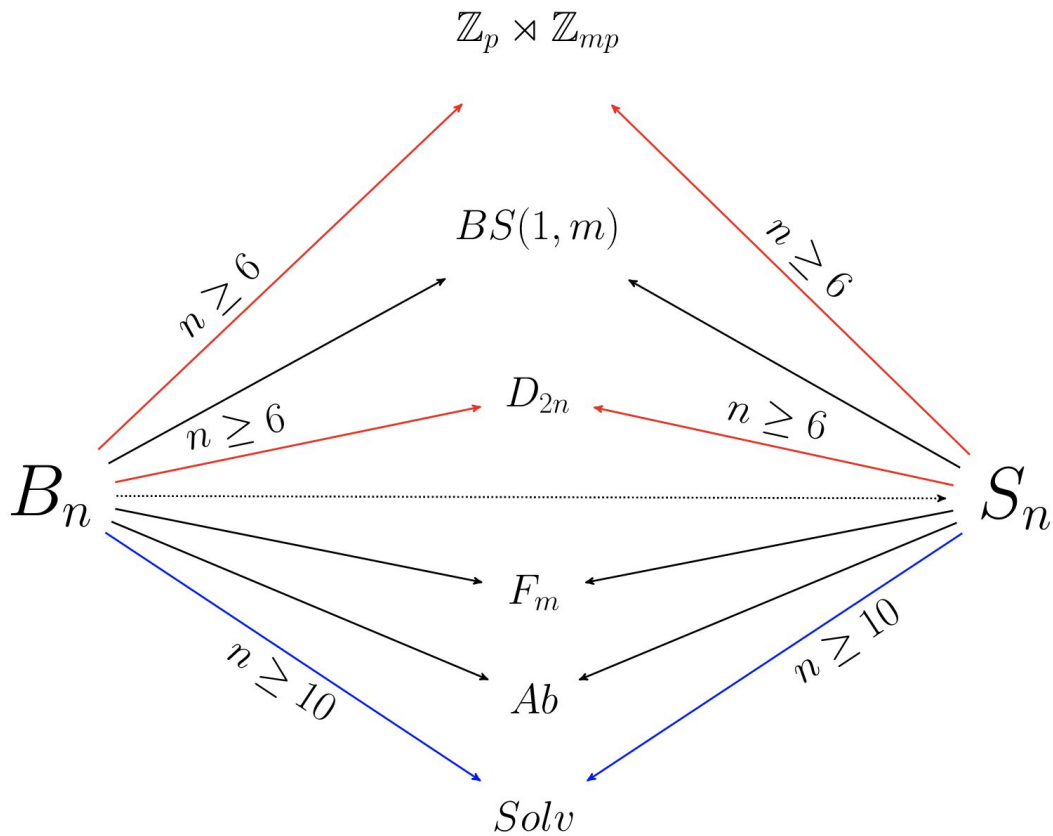
The image of S is either...

singleton

Then the map is
cyclic

$$\langle\langle\sigma_1\sigma_3^{-1}\rangle\rangle \cong B'_n$$

$$\langle\langle \sigma_1 \sigma_3^{-1} \rangle\rangle \cong B'_n$$



The image of S is either...

a singleton

not a singleton

Then the map is
cyclic

The stabilizer of S in G
Its image is large: contains a
copy of S_n

Case 2: Not singleton

$$B_n \rightarrow G$$

Case 2: Not singleton

$$\begin{array}{l} B_n \rightarrow G \\ \cup \\ \Gamma_S \rightarrow S_{\lfloor \frac{n}{2} \rfloor} \end{array}$$

Case 2: Not singleton

$$B_n \rightarrow G$$
$$\cup$$
$$\Gamma_S \rightarrow S_{\lfloor \frac{n}{2} \rfloor}$$

Stabilizer of the totally symmetric set

Case 2: Not singleton

$$B_n \rightarrow G$$

U

$$\Gamma_S \rightarrow S_{\lfloor \frac{n}{2} \rfloor}$$

The diagram illustrates the relationship between the groups B_n , G , Γ_S , and $S_{\lfloor \frac{n}{2} \rfloor}$. The mapping $B_n \rightarrow G$ is shown in a light gray font. Below it, a union symbol \cup is centered. The mapping $\Gamma_S \rightarrow S_{\lfloor \frac{n}{2} \rfloor}$ is shown in black. Two boxes with arrows provide context: one box labeled "Size of the totally symmetric set" has an arrow pointing to the $\lfloor \frac{n}{2} \rfloor$ term in the second mapping; another box labeled "Stabilizer of the totally symmetric set" has an arrow pointing to the Γ_S term in the second mapping.

Size of the totally symmetric set

Stabilizer of the totally symmetric set

Case 2: Not singleton

$$B_n \rightarrow G$$
$$\cup$$
$$1 \rightarrow K \rightarrow \Gamma_S \rightarrow \mathcal{S}_{\lfloor \frac{n}{2} \rfloor} \rightarrow 1$$

Elements inducing trivial permutation

All the elements that permute S

Case 2: Not singleton

$$\begin{array}{ccccccc} & & B_n & \rightarrow & G & & \\ & & & & \cup & & \\ 1 & \rightarrow & K & \rightarrow & \Gamma_S & \rightarrow & S_{\lfloor \frac{n}{2} \rfloor} \rightarrow 1 \end{array}$$

$$\boxed{|G| \geq |\Gamma_S| = |K| * |S_{\lfloor \frac{n}{2} \rfloor}|}$$

Bounds on Sizes of Totally Symmetric Sets

G	$S(G)$
F_n	1
D_{2n}	2
$\mathbb{Z}/np \rtimes \mathbb{Z}/p$	2
$BS(1, n)$	1 or 2
$SL_2(\mathbb{C})$	2

G	$S(G)$
B_n	$\lfloor \frac{n}{2} \rfloor$
S_n	$\geq \lfloor \frac{n}{2} \rfloor$
$Aut(F_n)$	$\geq n$

G	$S(G)$
$G \times H$	$\max(S(G), S(H))$
Ab	1
Odd	1
$Solv$	≤ 4