Puzzle on Graphs: Total Difference Labelings of Graphs

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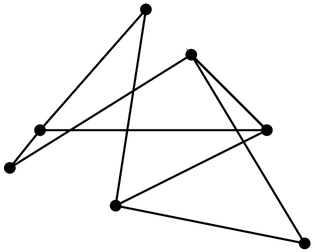
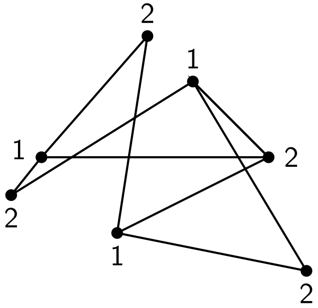
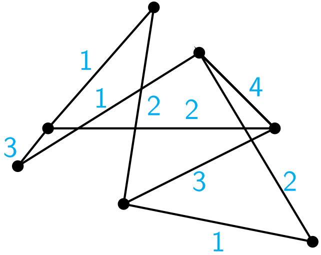
Graph



Proper Vertex Labeling and Proper Edge Labeling



Chromatic Number
$\chi(G) = 2$

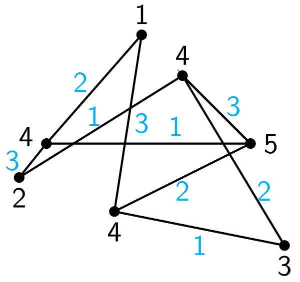


Chromatic Index
$\chi^{'}(G) = 3$

k-total labeling

A *total labeling* is an assignment of positive numbers to both vertices and edges, where

1. no two adjacent vertices share the same label,
2. no two incident edges share the same label,
3. and no incident edge and vertex share the same label.

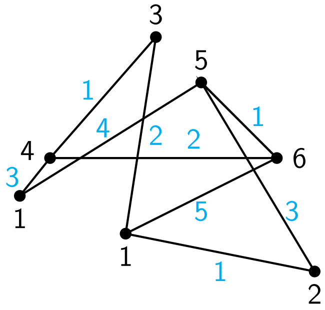


Total Chromatic Number
$\chi''(G) = 5$

k-graceful labeling

A *graceful labeling* is obtained by

1. properly label the set of vertices
2. label the edges by taking the absolute difference of incident vertex labels
3. where the set of edges are also properly labeled



Graceful Chromatic Number
$\chi_g(G) = 6$

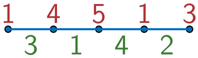
Total Difference Labeling

Let G be a graph.

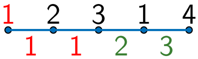
Steps:

1. Label the vertices with any number in the set of $\{1, 2, \ldots, k\}$.

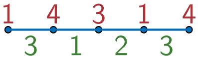
2. Label the edges with the absolute difference of end vertex labels.



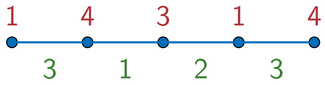
3. Make sure the labeling of G forms a total labeling.



4. Determine the smallest k such that G will be labeled this way. $(\chi_{td}(G))$

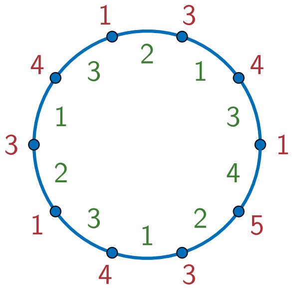


Paths and Cycles



P_5

$$\chi_{td}(P_n) = 4 \text{ for n} \geq 4$$



C_{10}

$$\chi_{td}(C_n) = \begin{cases} 4 & \text{if } n \equiv 0 \pmod 3 \\ 5 & \text{otherwise} \end{cases}$$

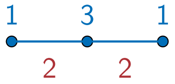
Total Difference Labeling

Let G be a graph.

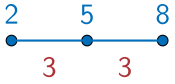
Steps:

1. Properly label the vertices with any number in the set of $\{1, 2, \ldots, k\}$, where the vertex labels do not contain doubles or 3-sequences.

$(2, 1)$-double



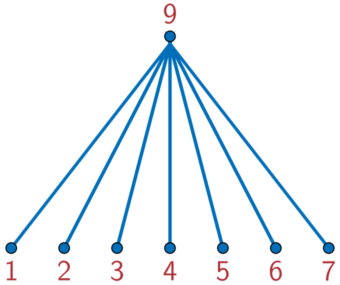
$(1, 3, 1)$-sequence



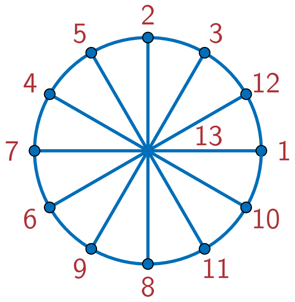
$(8, 5, 2)$-sequence

2. Determine the smallest k such that G will be labeled this way. $(\chi_{td}(G))$

Stars and Wheels



$K_{1,7}$

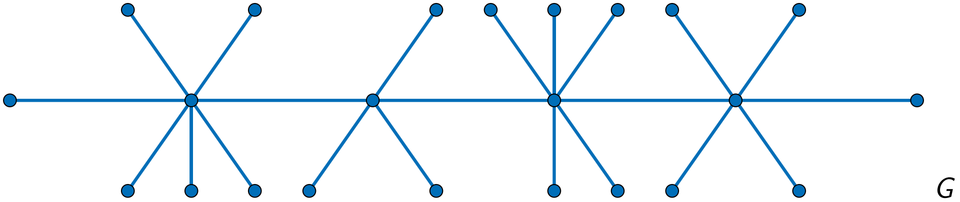


W_{13}

$$\chi_{td}(K_{1,m}) = \begin{cases} m+1, & m \text{ is even} \\ m+2, & m \text{ is odd} \end{cases}$$

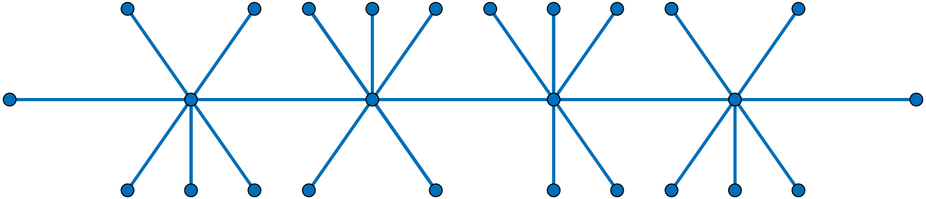
$$\chi_{td}(W_n) = \begin{cases} 8 & n = 4 \\ 7 & n = 5 \\ n+1 & n \text{ is even and } n \geq 6 \\ n & n \text{ is odd and } n \geq 7 \end{cases}$$

Caterpillars

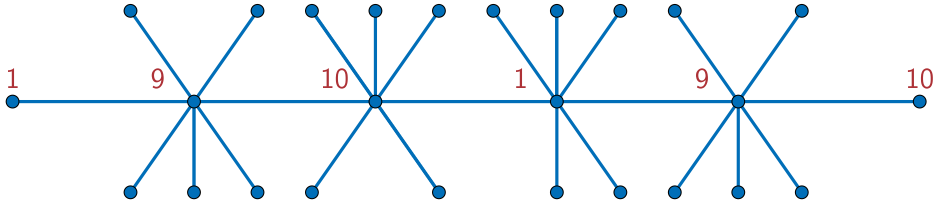


$$\Delta + 1 \leq \chi_{td}(G) \leq \Delta + 3$$

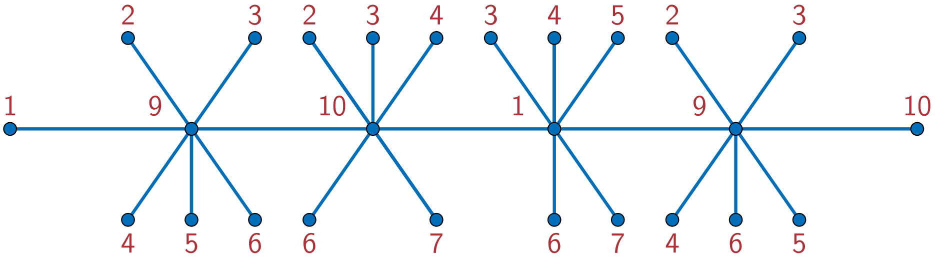
$\Delta + 3$



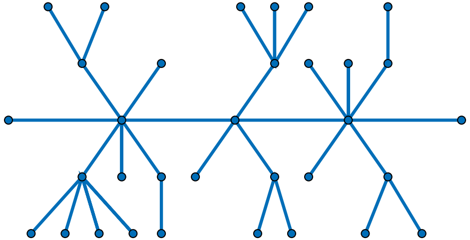
$\Delta + 3$



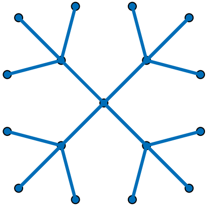
$\Delta + 3$



Lobsters and Maximal Rooted Trees



$$\Delta + 1 \leq \chi_{td}(H) \leq \Delta_1 + \Delta_2 + 1$$



For a maximal rooted tree with height 2,

$$\chi_{td}(T_{\Delta,2}) = \left\lfloor \frac{3\Delta + 3}{2} \right\rfloor.$$

For any maximal rooted tree with height h, where $h \geq 2$,

$$\left\lfloor \frac{3\Delta + 3}{2} \right\rfloor \leq \chi_{td}(T_{\Delta,h}) \leq 2\Delta + 1$$