

Isoperimetric Regions on the Real Number Line with Density $|x|$ and $||x| - 1|$

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Abstract

An isoperimetric-style problem is a classic problem of examining all geometric objects with a specified/fixed quantity, and finding the object that has an optimal quantity.

We seek to prove isoperimetric-style problems on spaces with density (that is, spaces with a function that effects how underlying geometric quantities such as length and area are measured). To this end, we examined the real number line \mathbb{R}^1 with two radially symmetric, eventually increasing densities: $|x|$ and $||x| - 1|$ (the "V" and "W" density, respectively). Our work resulted in three novel results:

1. A proof for the optimal single bubble on the W density.
2. A proof for the optimal double bubble on the W density.
3. A proof for the optimal triple bubble on the V density.

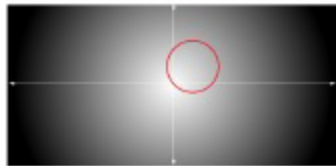
These results generalize the known single- and double-bubble results for the V density, discovered by Huang et. al.

The Classic Isoperimetric Problem

The classic isoperimetric problem is an ancient math problem that has been immortalized in literature as Queen Dido's Problem. In the Aeneid, a classic work of Roman literature, Queen Dido was granted land to form the city of Carthage, she used the ideas present in the isoperimetric problem to enclose as much land as possible in a rope made from ox-hide, and she used this land to form the city. Despite its origins over a millennium ago, the isoperimetric problem has only been rigorously solved in the past 200 years.

Prescribed Density on Ambient Space

Isoperimetric regions Ω exist in an ambient space on which geometric measurements (like area and perimeter) can be made. A prescribed density is a function f defined on the ambient space that changes the way area and perimeter are measured at each point in space. This in turn affects what regions might be considered isoperimetric regions.



Radially Increasing Density $f(x) = |x|$

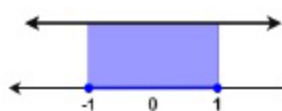


Uniform Density $f(x) = 1$

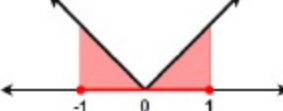
The two pictures above represent \mathbb{R}^2 imbued with two different densities. The right picture represents our usual (uniform) density $f \equiv 1$. In such a plane, area and perimeter are measured in the usual way, and so isoperimetric regions are circles that can appear anywhere. The left picture represents a radially increasing density function. In such a plane, area and perimeter are "heavier" as we move further away from the origin. Isoperimetric regions will still be circles, but they will take advantage of the low density at the origin and must touch the origin tangentially.

Imbuing the Real Number Line \mathbb{R}^1 with densities $|x|$ and $||x| - 1|$.

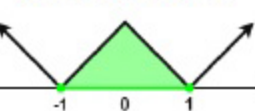
Number Line with density 1
Mass = 2 Perimeter = 2



Number line with density $|x|$
Mass = 1 Perimeter = 2



Number line with density $||x| - 1|$
Mass = 1 Perimeter = 0



The Single Bubble with Density $||x| - 1|$

Theorem 1: Suppose we have a region R with a given mass M and finite perimeter P , consisting of (possibly many) intervals on the line with density $||x| - 1|$. Then we can find a new region R' consisting of one interval $[-1, a]$ with mass M and perimeter P' with $P' \leq P$.



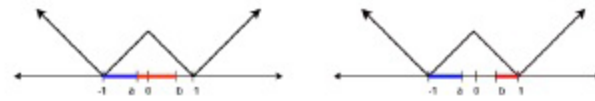
Key Elements of Proof

- **Proposition 1:** For the line with density $|x - a|$, a single interval region with one endpoint at a has less perimeter than a region of the same mass consisting of multiple intervals on the same side of a .
- **Proposition 4:** Suppose we have a region R that is a subset of $[-1, 1]$, with mass M and perimeter p . Then we can find a new region $R' = [-1, d]$ with the same mass M but with decreased perimeter p' .

The Double Bubble with Density $||x| - 1|$

Theorem 2a: Suppose we have two regions R_1 and R_2 (possibly consisting of many intervals) on the line of density $||x| - 1|$. Suppose our two regions have masses M_1 and M_2 , respectively, and total perimeter P . Assume that $M_1 < M_2$ and if $M_1 + M_2 \leq 1$. Then we can find two new regions R'_1 and R'_2 with masses M_1 and M_2 and new perimeter $P' \leq P$. Our new configuration will look like one of the following:

R'_1 will consist of one interval $[-1, a]$, and the position of R'_2 will depend on the size of M_2 . If $M_2 > \frac{1-M_1}{2}$, then R'_2 will consist of one interval $[a, b]$ and if $M_2 < \frac{1-M_1}{2}$, then R'_2 will consist of one interval $[b, 1]$.



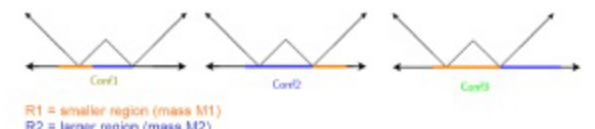
Theorem 2b: Suppose we have two regions R_1 and R_2 (possibly consisting of many intervals) on the line of density $||x| - 1|$. Suppose our two regions have masses M_1 and M_2 , respectively, and total perimeter P . Assume that $M_1 < M_2$ and $M_1 + M_2 > 1$, and assume that condition (*) does not hold. We can find two new regions R'_1 and R'_2 with masses M_1 and M_2 and new perimeter $P' \leq P$. The specific configurations are described in the table below.

	$0 < M_2 \leq 0.5$	$0.5 < M_2 \leq 1$	$1 < M_2 < \infty$
$0 < M_1 \leq 0.5$	Theorem 2a	Conf1	Conf2
$0.5 < M_1 \leq 1$	x	Conf1	(*)
$1 < M_1 < \infty$	x	x	Conf3

Configuration 1: $R'_2 \subseteq [-1, 1]$.

Configuration 2: $[-1, 1] \subseteq R'_2$.

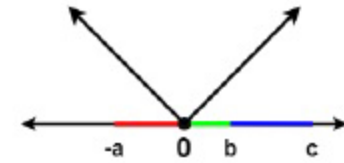
Configuration 3: $[-1, 1] \subseteq R'_1$.



R'_1 = smaller region (mass M_1)
 R'_2 = larger region (mass M_2)

The Triple Bubble on Density $|x|$

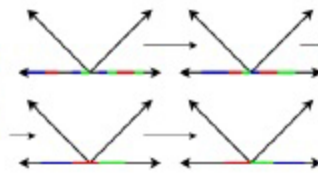
Theorem 3: Suppose we have three regions R_1, R_2 , and R_3 (possibly consisting of many intervals) on the line of density $|x|$. Suppose our three regions have masses M_1, M_2 , and M_3 , respectively, and total perimeter P . In addition, assume that $M_1 \leq M_2 \leq M_3$. We can find three new regions R'_1, R'_2 , and R'_3 with masses M_1, M_2 , and M_3 and new perimeter $P' \leq P$. Here, R'_1 will consist of one interval $[0, b]$, R'_2 will consist of one interval $[-a, 0]$, and R'_3 will consist of one interval $[b, c]$.



The least-perimeter arrangement of three regions has the two smaller regions sharing the origin as an endpoint, much like in the double bubble case.

Key Elements of Proof and Challenges

1. By Proposition 1.1, we can gather all intervals into no more than three single-interval regions on each side of the origin.
2. By Proposition 1.2, we can combine intervals across the origin so each region is only one interval.
3. By Propositions 1.4, 1.6, and 1.3 in conjunction with 1.5, arranging the three regions with one adjacent to one side of the origin and the other two adjacent on the other side requires equal or less perimeter than any other arrangement of the three regions.
4. By Proposition 1.7, the least-perimeter way to order three regions by mass in the one-origin-two style is with the two smallest-mass regions adjacent to the origin and the largest region adjacent to the smallest region.



Future Problems

After finding the solution to the triple bubble problem on density $|x|$, it is natural to look past that to four, five, or even n -bubble problems. We do not have definitive proof about the least-perimeter configuration of n regions, but we propose that it follows a similar pattern to the triple bubble, with the smallest regions adjacent to the origin and progressively larger regions alternating across the origin adjacent to the others.



A proposed least-perimeter arrangement, where $M_1 \leq M_2 \leq \dots \leq M_{n-1} \leq M_n$.

Acknowledgements

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References

- [1] Vincent Bayle, Frank Morgan, Cesar Rosales, Antonio Canete. On the isoperimetric problem in euclidean space with density. Calculus of Variations, 31, 2008.
- [2] Juiyu Huang et. al. Isoperimetric problems on the line with density $|x|^p$. Rose-Hulman Undergraduate Mathematics Journal, 20, 2019.