The Game of Cycles: Extended

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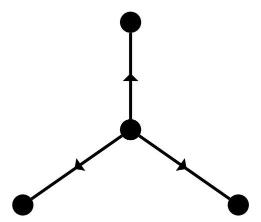
Motivation

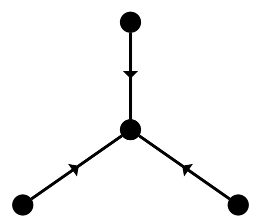
Graph theory is a field of mathematics that can be utilized to model a variety of processes and thus knowing more about them increases our ability to solve a broad range of applications.

Definitions

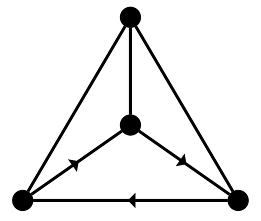
- Planar graph: A collection of vertices and edges such that no two edges cross
- Directed graph: A graph with arrows drawn on some or all edges indicating direction
- Sink: A vertex of a directed graph with all edges connected to that vertex directed towards that vertex
- Source: A vertex of a directed graph with all edges connected to that vertex directed away from that vertex
- Cycle cell: An enclosed area of a graph with nothing inside of it such that every edge enclosing the area is directed the same direction



Source

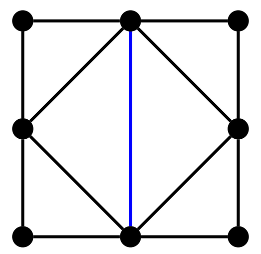


Sink



cycle cell

- The Game of Cycles: A game with two or more players where on each turn a player directs one edge of a graph and does not create a sink or source. The object of the game is to produce a cycle cell.
  - The first player to create a cycle cell wins or the last player to move wins.
- Involutive Symmetry: A board has involutive symmetry if there is a non-trivial symmetry τ of the board which is its own inverse. τ assigns a unique partner to each vertex, edge, and cell and is called an involution
- Nowhere-involutive edge: An edge where no edge of the cell has its partner in the cell from the involution
- Self-involutive edge: An edge where the involution of an edge is itself



Graph with a self-involutive edge.

Goal

Our research team aims to answer questions about the Game of Cycles on various families of planar graphs.

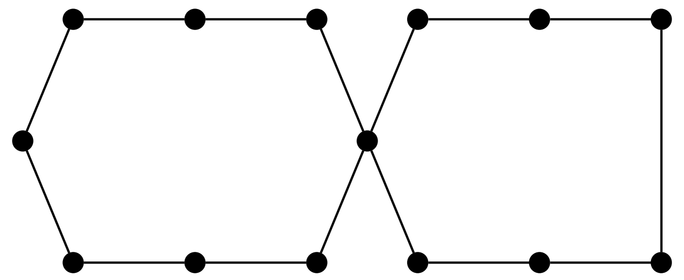
Related Theorems

Theorem 6: [2] Let G be a board with an involution such that each cell is either self-involutive or nowhere-involutive. If there is no self-involutive edge, then Player 2 has a winning strategy. If there is exactly one self-involutive edge whose vertices are not fixed by the involution, then Player 1 has a winning strategy using the mirror-reverse strategy.

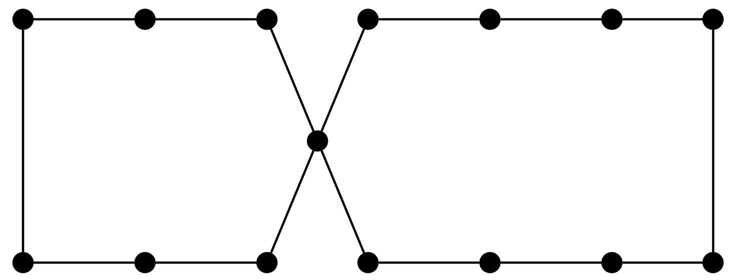
Mirror-Reverse Strategy

- If possible to win by completing a cycle, do so.
- If that is not possible, mirror other player's strategy by observing the player's most recent move $i \to j$ and playing $j' \to i'$, the partner edge of ij with its arrow reversed.

Main Results



Cycle Graphs C_8 and C_7



Cycle Graphs C_7 and C_9

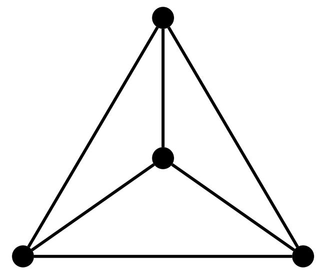
Theorem: Let G be a board containing two cycles, C_m and C_n, such that C_m and C_n are connected by a single vertex. If $m = 3$ and $n = 3$, then Player 2 has a winning strategy. If $m = 3$ and $n > 3$, then Player 1 has a winning strategy.

Theorem: Let G be a board containing two cycles, C_m and C_n, such that C_m and C_n are connected by a single vertex. If $m > 3$ and $n > 3$ and of the same parity, then Player 2 has a winning strategy.

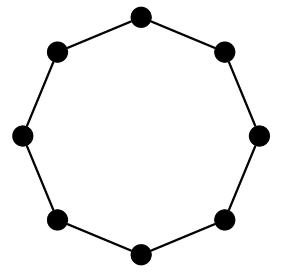
Theorem: Let G be a board containing two cycles, C_m and C_n, such that C_m and C_n are connected by a single vertex. If $m > 3$ and $n > 3$ and are of different parity, then Player 1 has a winning strategy by the mirror-reverse strategy.

Theorem: Let G be a board containing three cycles, C_m, C_n, and C_p such that C_m, C_n and C_p are connected by a single vertex. Let m, n, and p be greater than 3 and odd, then Player 1 has a winning strategy by an alternative of the mirror-reverse strategy.

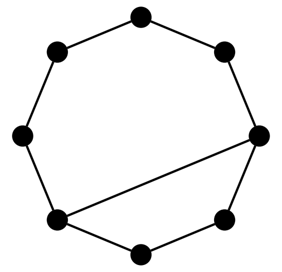
Related Results



Complete Graph K_4



Cycle Graph C_8



Cycle Graph with a chord

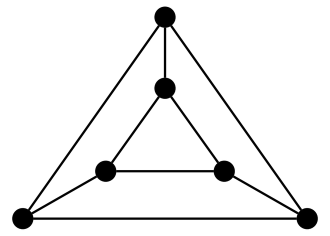
Theorem: [2] On a K_4 board, Player 2 has a winning strategy.

Theorem: [2] The play on a C_n board is entirely determined by parity. If n is odd, Player 1 wins. If n is even, Player 2 wins.

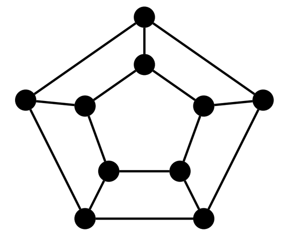
Theorem: [2] Let $n \geqslant 4$. Consider a C_n board with a chord. If n is even, then Player 1 has a winning strategy, and if n is odd, then Player 2 has a winning strategy.

Other Results

Theorem: In a two player game played on the 3-Prism and 5-Prism graphs, Player 1 has a winning strategy.



3-Prism Graph



5-Prism Graph

A computer program determined the strategy of Player 1.

Theorem: Let C_n be the cycle graph on n vertices and i be the number of pairs of unmarkable edges. Then $i \leqslant \lfloor \frac{n}{4} \rfloor$.

Theorem: Consider a multi-player game of p on the cycle board C_n with n vertices. Let i be the number of pairs of unmarkable edges. Upon completion of the game, if $n - 2i \equiv m \mod p$, then Player m wins.

Future Research

- Can we determine an overarching strategy for m connected cycle graphs comprised of n odd cycles and p even cycles?
- If $n > 5$ is odd, in a two player game, which player will win if played on the n-Prism graph?

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