Robust and Efficient Phase Retrieval from Magnitude-Only Windowed Fourier Measurements

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Motivating Application – Molecular Imaging

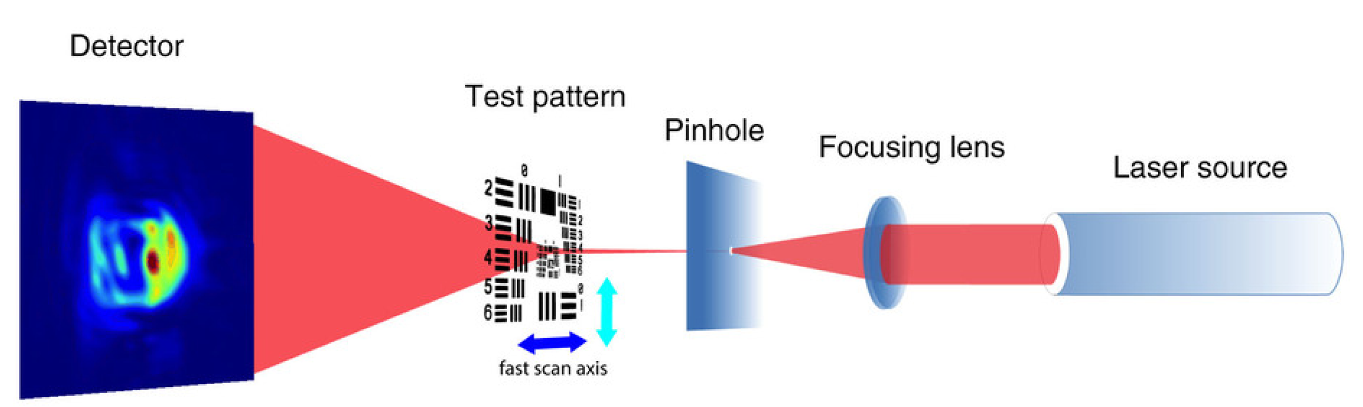


Fig.: Image source: Xiaojing, et al. "Fly-scan ptychography." Scientific Reports 5 (2015).

The phase retrieval problem occurs in several fields of science such as X-ray crystallography, optics, astronomy and quantum mechanics, where, either due to the underlying physics or instrumentation limitations, we are unable to acquire phase information.

Acquired Measurements (Spectrogram Model)

$$\left|\sum_{n=0}^{d-1} x_n m_{n-l} e^{\frac{-2\pi i n j}{d}}\right|^2 = Y_{j,l}$$

Unknown signal/specimen is $\mathbf{x} \in \mathbb{C}^d$; $\mathbf{m} \in \mathbb{C}^d$ is a *known* mask/window (models pinhole and focusing lens); l denotes a (discrete) shift of the mask

Understanding Phase Retrieval – A 2×2 Model Problem

Suppose we have the magnitude-only (phaseless) system of equations

$$|2ix + 4y|^2 = |6|^2 = 36$$
$$|3x - iy|^2 = |i|^2 = 1$$

Lifting/Linearization:

$$\begin{bmatrix} 4 & 8i & -8i & 16 \\ 9 & 3i & -3i & 1 \\ 4 & 6 & 6 & 9 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} |x|^2 \\ x\overline{y} \\ \overline{x}y \\ |y|^2 \end{bmatrix} = \begin{bmatrix} 36 \\ 1 \\ 40 \\ 16 \end{bmatrix} \xrightarrow[\text{*additional (given) equations}]{\text{new unknown}} \mathbf{x} = \left[|x|^2 \; x\overline{y} \; \overline{x}y \; |y|^2\right]^T$$

Angular/Vector Synchronization: Having solved for the vector $\mathbf{x} = \left[|x|^2 \; \overline{x}y \; x\overline{y} \; |y|^2\right]^T$, we obtain x and y as follows

$$\begin{bmatrix} |x|^2 \\ \overline{x}y \\ x\overline{y} \\ |y|^2 \end{bmatrix} \xrightarrow{\text{re-arrange}} \begin{bmatrix} |x|^2 & x\overline{y} \\ \overline{x}y & |y|^2 \end{bmatrix} \xrightarrow{\text{eigenvector}} \begin{bmatrix} x \\ y \end{bmatrix}.$$

How can we apply the 2 x 2 Model Problem to linearize and perform vector/angular synchronization on the Spectrogram Model?

Proposed Framework

We linearize the problem, using the following theorem, where $F_d : \mathbb{C}^d \to \mathbb{C}^d$ is the discrete Fourier Transform (DFT), $S_\alpha : \mathbb{C}^d \to \mathbb{C}^d$, $(S_\alpha \mathbf{x})_n := x_{n-\alpha}$, $n, \alpha \in \mathbb{Z}_d$ is the (circular) shift operator, $\circ, \overline{\mathbf{x}}, M^*$ denote component-wise multiplication, complex conjugate and conjugate transpose respectively.

Theorem (Perlmutter et al.)

Let $\mathbf{x} \in \mathbb{C}^d$ be an unknown d-periodic signal and let $\mathbf{m} \in \mathbb{C}^d$ be a known *mask*. Consider the measurement model

$$Y_{k,l} = \left|\sum_{n=0}^{d-1} x_n m_{n-l} e^{\frac{-2\pi i n k}{d}}\right|^2 = |(F_d(\mathbf{x} \circ S_l \mathbf{m}))_k|^2 \qquad k, l \in \mathbb{Z}_d,$$

where $Y_{k,l}$ denotes acquired measurements (corresponding to the k^{th} frequency index and l^{th} shift). Define the sub-sampled measurement matrix $Y_{K,L} \in \mathbb{R}^{K \times L}$ (for some $K \mid d, L \mid d$) as

$$(Y_{K,L})_{k,l} := Y_{\frac{kd}{K}, \frac{ld}{L}}, \qquad k \in \mathbb{Z}_K, \; l \in \mathbb{Z}_L.$$

Let $\widetilde{Y} := F_L Y_{K,L}^T F_K^T$ Then, for $\omega \in \mathbb{Z}_K$ and $\alpha \in \mathbb{Z}_L$,

$$\underbrace{\widetilde{Y}_{\alpha,\omega}}_{\text{2 FTs of measurements}} = \frac{KL}{d} \sum_{r=0}^{\frac{d}{K}-1} \sum_{l=0}^{\frac{d}{L}-1} (F_d \underbrace{(\mathbf{x} \circ S_{\omega - rK} \overline{\mathbf{x}})}_{\text{unknown signal terms}})_{\alpha - lL} (F_d \underbrace{(\mathbf{m} \circ S_{\omega - rK} \overline{\mathbf{m}})}_{\text{known mask terms}})_{lL - \alpha}.$$

We then perform angular synchronization.

$$\text{Thm.} \xrightarrow[\text{linear system}]{\text{solve}} \begin{bmatrix} \underbrace{\mathbf{x} \circ S_{-(\kappa-1)} \overline{\mathbf{x}}}_{d\text{-(row) vector}} \\ \cdots \\ \mathbf{x} \circ S_{-1} \overline{\mathbf{x}} \\ \mathbf{x} \circ S_0 \overline{\mathbf{x}} \\ \mathbf{x} \circ S_1 \overline{\mathbf{x}} \\ \cdots \\ \mathbf{x} \circ S_{(\kappa-1)} \overline{\mathbf{x}} \end{bmatrix}_{\in \mathbb{C}^{(2\kappa-1) \times d}} \xrightarrow[\substack{2\kappa - 1 \\ \text{diagonals}}]{\text{re-arrange along}} T_\kappa(\mathbf{x}\mathbf{x}^*), \quad (T_\kappa(Z))_{j,k} = \begin{cases} Z_{j,k}, & |j-k| \bmod d < \kappa \\ 0, & \text{else.} \end{cases}$$

This produces a Hermitian symmetric banded matrix such that the main diagonal contains squared magnitudes of $\mathbf{x}$ and off-diagonals contain relative phase information.

Properties of T_κ

- The largest eigenvalue of T_κ is $2\kappa - 1$, with corresponding eigenvector $\tilde{\mathbf{x}}$ where $(\tilde{\mathbf{x}})_j = e^{i\phi_j}$, and $\phi_j \in [0, 2\pi]$ such that $\frac{x_j}{|x_j|} = e^{i\phi_j}$. The unknown phases of the signal are contained in $\tilde{\mathbf{x}}$.
- The largest eigenvalue is unique. For the matrix T_κ, if the eigenvalues are ordered such that λ_1 is the largest, then $\lambda_1 \neq \lambda_2$.

Theoretical Error Bound

For $\mathbf{x}, \mathbf{m} \in \mathbb{C}^d$ with $\text{supp}(\mathbf{m}) \subseteq \mathbb{Z}_\delta$ where $\delta < d/2$, let K divide d and $Y_{K,d}$ be the partial measurement matrix, with $N_{K,d}$ being the corresponding noise matrix. With $\widetilde{Y} := F_d Y_{K,d}^T F_K^T$ and $\widetilde{N} := F_d N_{K,d}^T F_K^T$, let $\min |\mathbf{x}| := \min_{0 \le n \le d-1} |x_n| > 0$ and let μ be a mask-dependent constant with $\mu := \min_{|p| \le \kappa-1, |q| \le \delta-1} \left|(F_d(\mathbf{m} \circ S_p \overline{\mathbf{m}}))_q\right| > 0$. Let $\kappa \in [2, \delta]$ and assume $K = \delta + \kappa - 1$ divides d. The reconstruction algorithm based on the proposed framework will output an estimate of $\mathbf{x}$, $\mathbf{x}_e$, such that

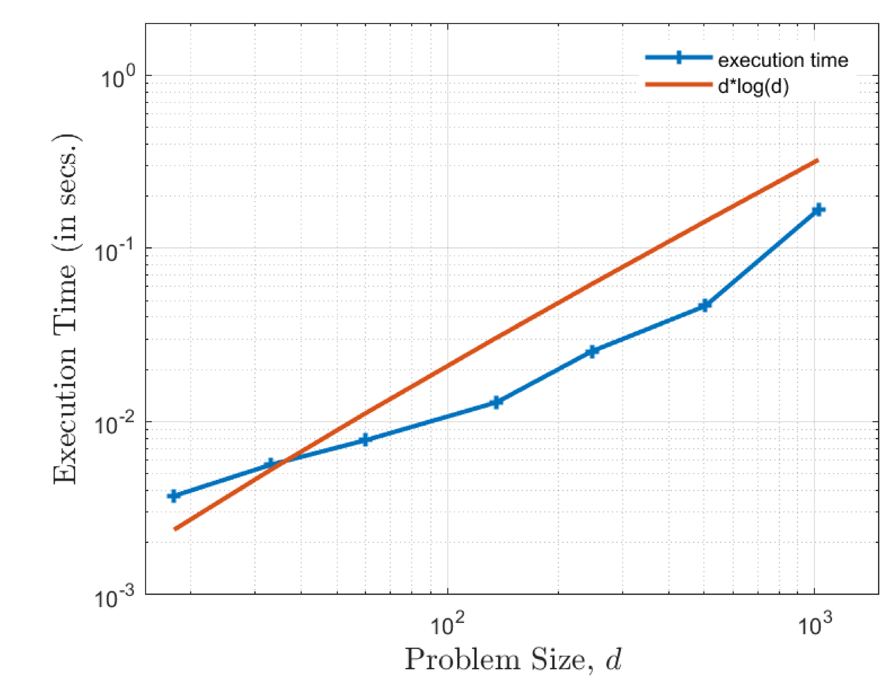
$$\min_{\phi \in [0, 2\pi]} ||\mathbf{x} - e^{i\phi}\mathbf{x}_e||_2 \le \frac{C \; ||N_{K,d}||_F \; d^2 \; ||\mathbf{x}||_\infty}{\kappa^2 \mu \sqrt{K(2\kappa - 1)} \min |\mathbf{x}|^2} + C' \sqrt{\frac{d}{\mu\sqrt{K}}} ||N_{K,d}||_F$$

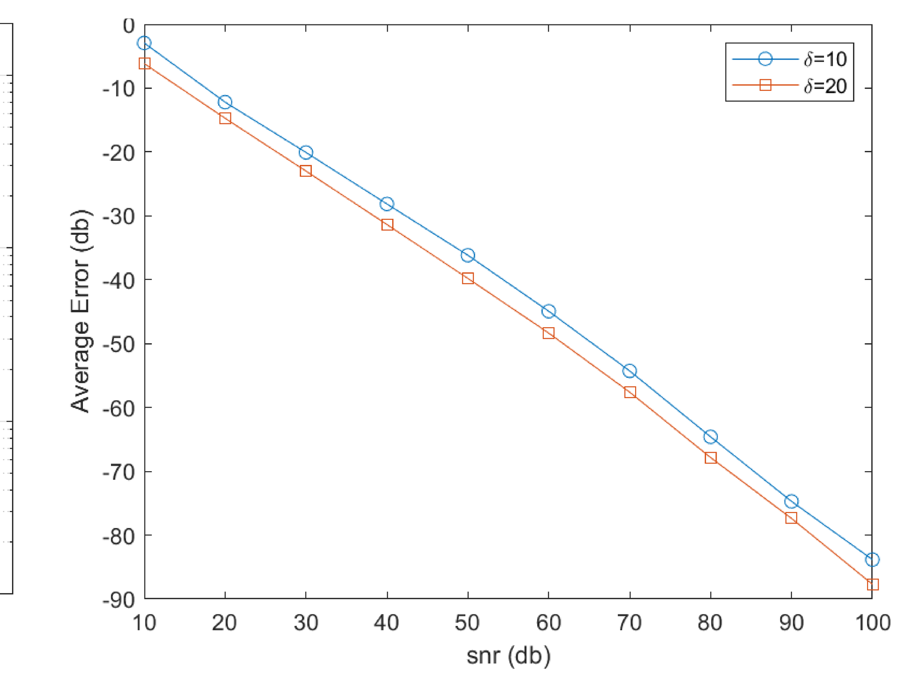
where C, C' are constants independent of all other factors.

Numerical Results

The left figure shows execution time (Matlab algorithmic implementation) as a function of problem size, which is in line with what we expect from the algorithm and use of the fast fourier transform.

The right figure illustrates robustness of the method to added noise – with both added noise (expressed as signal to noise ratio (SNR)) and reconstruction error measured in decibels (db) – which is in line with our predictions from the theoretical error bound.





References and Acknowledgement

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- *Inverting Spectrogram Measurements via Aliased Wigner Distribution Deconvolution and Angular Synchronization*, M. Perlmutter et al., Information & Inference, Oct 2020, in press

This research was completed at the REU Site: Mathematical Analysis and Applications at the University of Michigan-Dearborn. We would like to thank the NSF (DMS-1950102), NSA (H98230-19), and University of Michigan for their support. Thank you to our mentor Aditya Viswanathan.