An Introduction to Parking Functions

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ABSTRACT

In 1966, Alan G. Konheim and Benjamin Weiss defined "parking functions." In 1974, Henry O. Pollak proved the total number of parking functions of length n, meaning there are n parking spots and n cars, to be $(n + 1)$^$(n - 1)$. We describe a recursive formula, expound Pollak's succinct six-sentence proof of an explicit formula, and conclude with a discussion of other parking function generalizations.

DEFINITIONS

We have n cars and n parking spots on our one-way, one lane street. Each car c_i has a preferred parking spot, which we call its **preference** a_i.

Each car's preference α_i is listed in a **preference vector** $\alpha = (a_1, a_2, \ldots, a_n)$.

The **parking rule** is as follows:

- Each car drives to its preference.
- If car c_i's preference is available, it parks in its preferred spot.
- If car c_i's preference is occupied, it parks in the next available spot.
- If a preference vector α allows all cars to park, we call α a **parking function** of length n.

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Let PF_n denote the set of all parking functions of length n. Let $|PF_n|$ denote the number of parking functions of length n, the size of the set PF_n.

EXAMPLES

Consider (1,1,1,1).

Consider (2,3,3,4).

Claim: All permutations are parking functions.

Corollary: For $n > 1$, $|PF_n| > n!$

RECURSIVE FORMULA

As stated in [1], the number of parking functions of length n follows the recursive formula

$$|PF_n| = \sum_{i=1}^{n} i \binom{n-1}{i-1} |PF_{i-1}| \cdot |PF_{n-i}|.$$

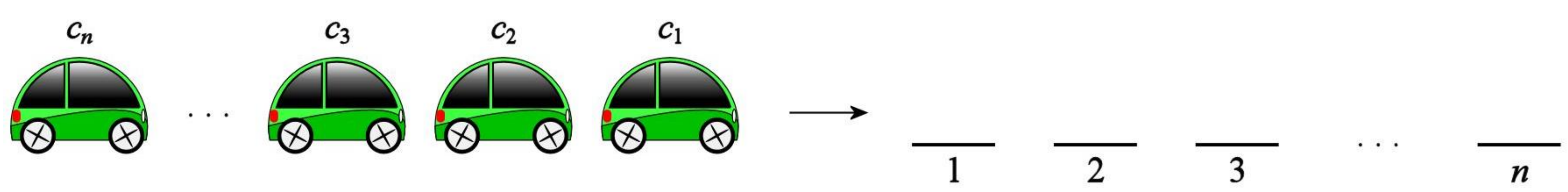

Figure 1 ***(above)***
Parking Function Illustration

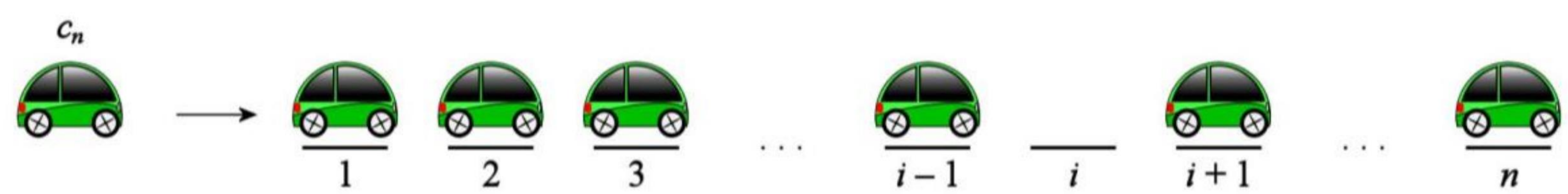

Figure 2 ***(above)***
Parking Spots Enumerated for Recursion

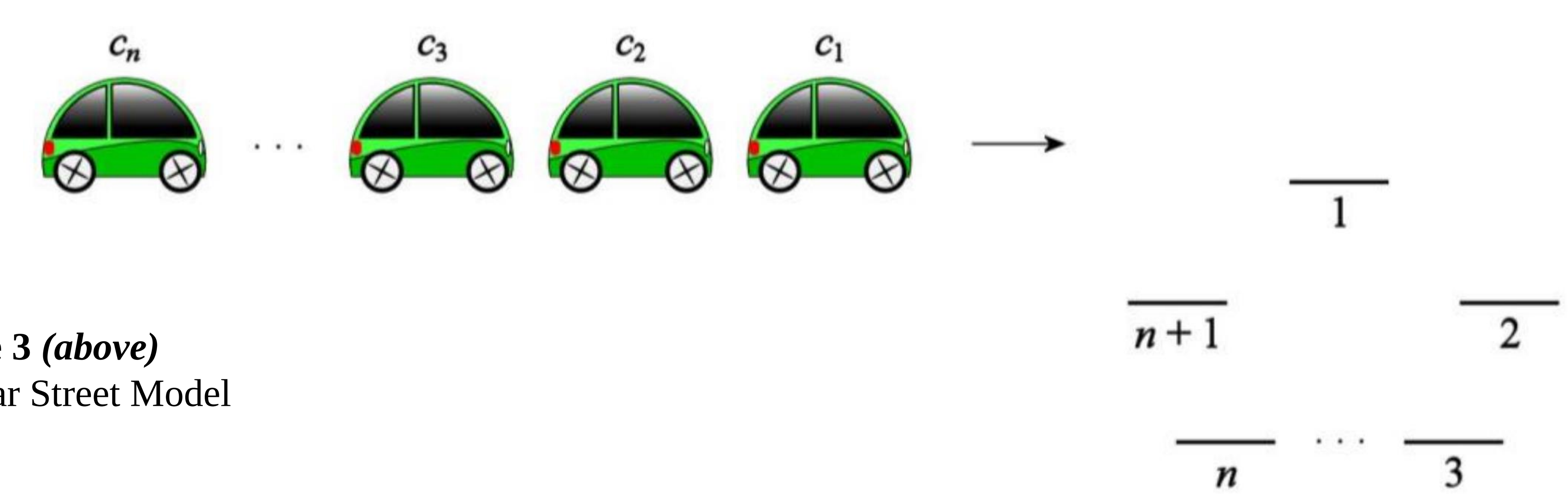

Figure 3 ***(above)***
Circular Street Model

Lemma for Sentence 5. Let $\alpha = (a_1, a_2, \ldots, a_n)$ be a parking function for a circular street with $n + 1$ parking spaces in which cars park in the following order $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ where the p_i are all distinct. Define $\alpha + 1$ to be the vector $(a_1 + 1, a_2 + 1, \ldots, a_n + 1)$ (modulo $n + 1$). The preference vector $\alpha + 1$, results in the cars parking in order $\boldsymbol{p+1} = (p_1 + 1, p_2 + 1, \ldots, p_n + 1)$.

Sketch of Proof: Proceed by contradiction and cases.

Example for Sentence 6 ***(below)***

c_i	$a_i = a_i + 0$	$p_i = p_i + 0$	$a_i + 1$	$p_i + 1$	$a_i + 2$	$p_i + 2$	$a_i + 3$	$p_i + 3$
1	1	1	2	2	3	3	0	0
2	2	2	3	3	0	0	1	1
3	2	3	3	0	0	1	1	2

EXPLICIT FORMULA

Proof as presented in [2].

"Add an additional space $n + 1$ and arrange the spaces in a circle.

Allow $n + 1$ also as a preferred space.

Now all cars can park, and there will be one empty space.

α is a parking function if and only if the empty space is $n + 1$.

If $\alpha = (a_1, a_2, \ldots, a_n)$ leads to car c_i parking at space p_i, then $\alpha + j = (a_1 + j, a_2 + j, \ldots, a_n + j)$ (modulo $n + 1$) will lead to car c_i parking at space $p_i + j$. [See Lemma for Sentence 5.]

Hence, exactly one of the vectors $(a_1 + k, a_2 + k, \ldots, a_n + k)$ (modulo $n + 1$) is a parking function, so

$$|PF_n| = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}.\text{"} \quad \square$$

GENERALIZATIONS

- Parking Completions
- Interval Parking Functions
- k-Naples Parking Functions

REFERENCES

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