Strategies and Generalizations of the Explorer-Director Game

The Explorer-Director Game

Two players control a token on a graph G, starting at a vertex v. Each round, **Explorer** calls a distance, then **Director** moves the token to a vertex that distance away.

- **Explorer** wants to *maximize* the number of vertices visited.
- **Director** wants to *minimize* the number of vertices visited.

$f_d(G, v)$ is the number of unique vertices visited on a graph G during an optimally played game starting at vertex v.

Definitions

Let $G = (V, E)$ be a graph.

- The **distance** between two vertices $u, v \in V$ is the length of the shortest path between them.
- The **eccentricity** of a vertex $v \in V$ is the maximum distance between v and any other vertex $u \in V$.
- The **radius** of G is the minimum eccentricity over $v \in V$.
- The **diameter** of G is the maximum eccentricity over $v \in V$.
- A set of vertices $A \subset V$ is **Director-closed** if when the token is at any vertex in the set, regardless of the distance Explorer calls, Director can keep the token in the set.
- A set of vertices $A \subset V$ is **Explorer-friendly** if no matter where the token is, Explorer has a strategy that will force the token into A regardless of Director's choices.

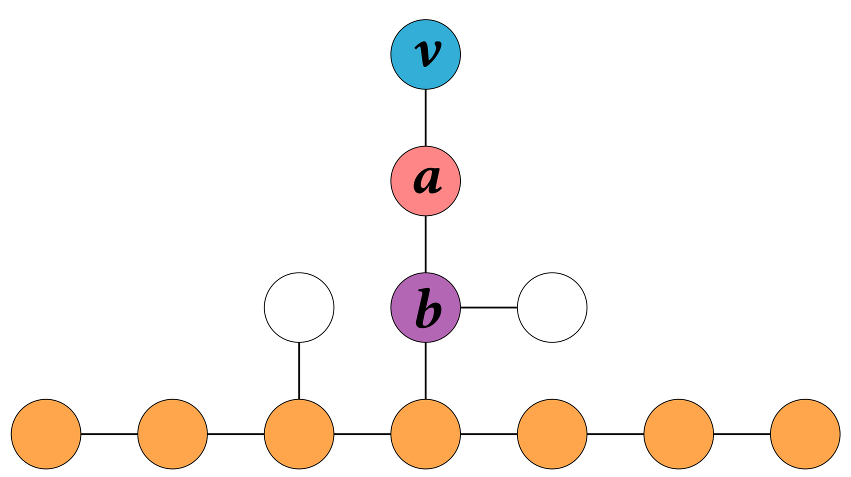
Trees

Let G be a tree. For a vertex v, let $diamDist(v)$ be the shortest distance from v to a longest path P of G (a diameter of G). Then

$$f_d(G, v) = diamDist(v) + diam(G) + 1$$

***P* is Director-closed**, so Director's strategy is to get to P as soon as possible, while Explorer's strategy is to stay away from P.

Example:



1. Explorer calls distance 1 to visit the vertex a adjacent to v.
2. Explorer calls 1 so Director must visit b or v. If Director visits v, Explorer then calls 2 to force the token to b.
3. Now Director can force the token to the diameter for any distance. All vertices along the diameter are visited like a path graph.

General Results

For any graph $G = (V, E)$ with vertex $v \in V$,

- At the end of the game, the set of visited vertices always *contains* a Director-closed set. (not necessarily the entire set of visited vertices!)
- **Lemma of Eccentricity:** Let $A \subset V$ be Explorer-friendly. Then $f_d(G, v) \geq \min_{u \in A} ecc(u) + 1$.
- $f_d(G, v) \geq radius(G) + 1 \geq \frac{diam(G)}{2} + 1$.
- $f_d(G, v) \leq |B|$, where B is any Director-closed subset of V *containing* v.

Paths

For a path graph P_n with n vertices and any starting vertex v,

$$f_d(P_n, v) = n.$$

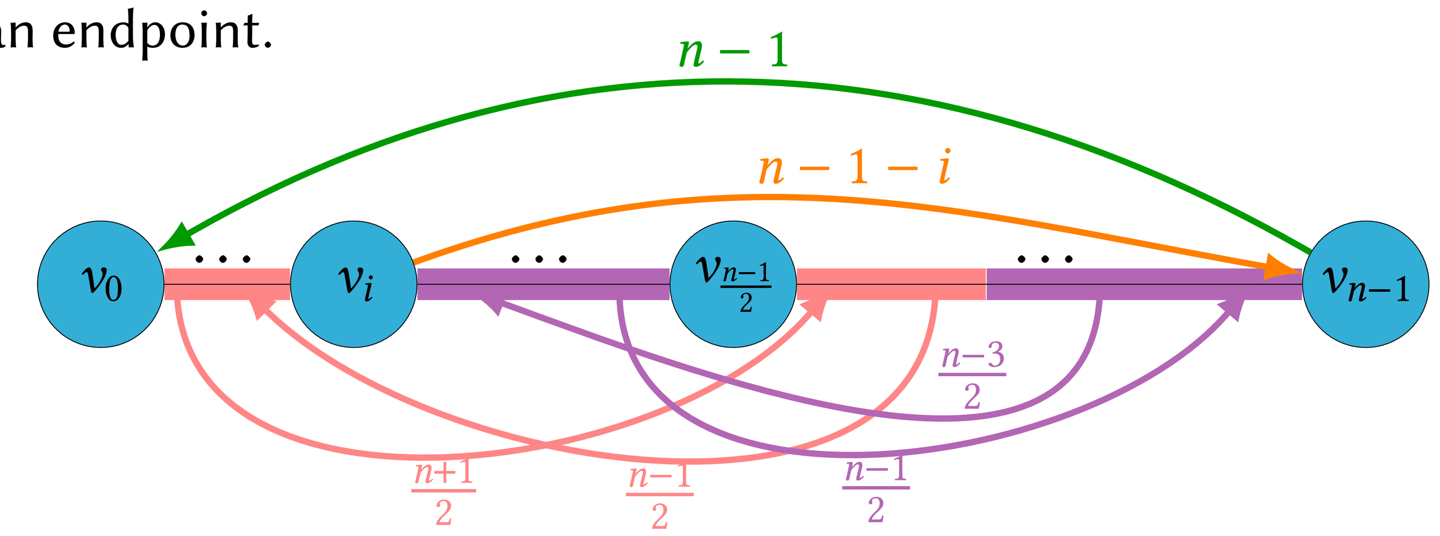
Non-Adaptive Explorer Strategies consist of a predetermined list of distances Explorer calls each round.

For path graphs, non-adaptive Explorer strategies can always be used to visit all n vertices in $n - 1$ rounds.

Characteristics of these strategies:

- Explorer calls large distances.
- For paths with an odd number of vertices, unless the token starts at the midpoint, it visits the midpoint last.
- Each round, the token visits a previously unvisited vertex.

Example: Below is the non-adaptive Explorer strategy for odd paths starting at a vertex that is not an endpoint or adjacent to an endpoint.



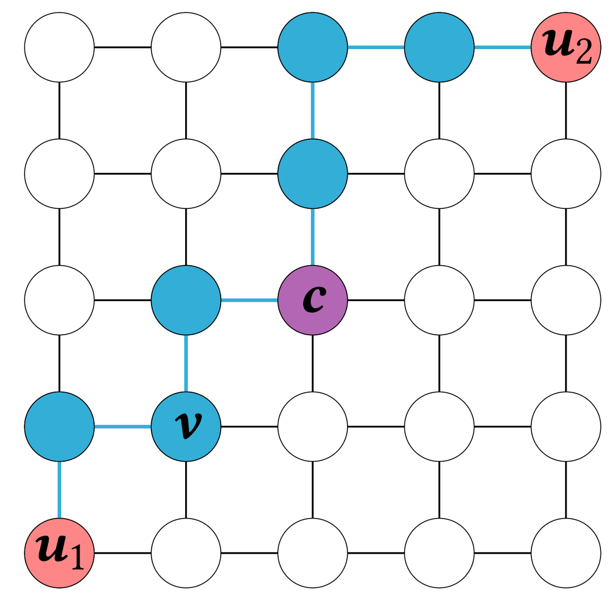
1. Explorer calls $n - 1 - i$ to visit the furthest endpoint.
2. Explorer calls $n - 1$ to visit the opposite endpoint.
3. Explorer alternately calls $\frac{n+1}{2}$ and $\frac{n-1}{2}$ for $2i - 1$ rounds, stopping just before they revisit v_i.
4. Explorer alternately calls $\frac{n-3}{2}$ and $\frac{n-1}{2}$ for the remaining $n - 2i - 2$ rounds to visit all remaining vertices.

Note: There is an *adaptive* Explorer strategy that will visit all vertices in $n - 1$ rounds that depends only on n, not v.

Lattices

Let G be an n by n square lattice where n is odd. Then

$$f_d(G, v) = 2n - 1 \quad \text{for any } v$$



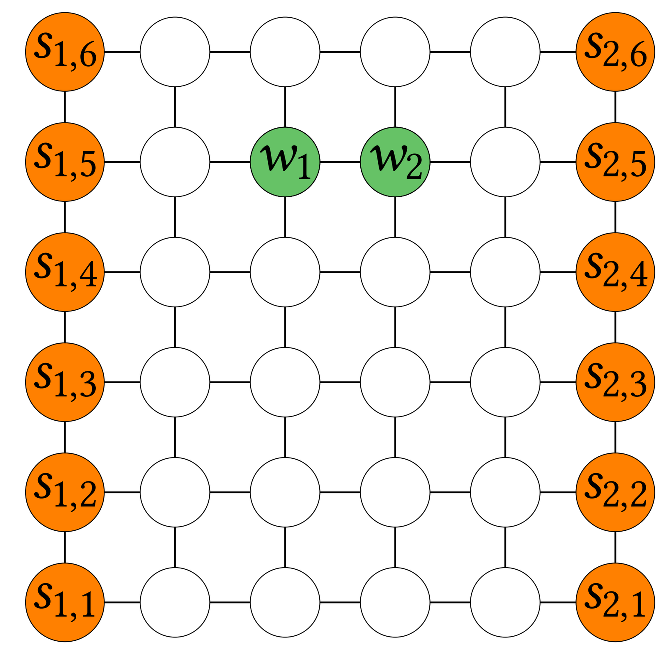
Director can find a path P of length $2n - 1$ through

- v, any starting vertex
- the center vertex c
- the corner closest to v, u_1, and its opposite corner u_2

***P* is Director-closed**
$\implies f_d(G, v) \leq 2n - 1$

Two opposite corners are Explorer-friendly since Explorer can always visit a corner u_i by calling $ecc(v)$ for $v \in P$. Since $ecc(u_i) = 2n - 2$, then by the Lemma of Eccentricity, $f_d(G, v) \geq 2n - 1$. Thus, $f_d(G, v) = 2n - 1$.

When n is even, P does not exist since there is no center vertex. Each vertex has *exactly one* farthest vertex.



Director can find a **Director-closed** set A of $3n - 4$ vertices consisting of

- two opposite sides $s_{i,j}$
- second row vertices non-adjacent to the opposite sides w_k

$\implies f_d(G, v) \leq 3n - 4$ for $v \in A$

Future Questions

- What is $f_d(G, v)$ for other types of graphs?
- For what other types of graphs do non-adaptive strategies allow Explorer to reach $f_d(G, v)$ vertices quickly?
- How can we efficiently identify which subsets of a given graph are Director-closed and/or Explorer-friendly?

References

Nedev, Z., & Muthukrishnan, S. (2008). The Magnus-Derek Game. *Theoretical Computer Science, 393*, 124-132.

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