The Explorer-Director Game

Two players control a token on a graph $G$, starting at a vertex $v$. Each round, Explorer calls a distance, then Director moves the token to a vertex that distance away. 

- Explorer wants to maximize the number of vertices visited. 
- Director wants to minimize the number of vertices visited. 

$f_d(G, v)$ is the number of unique vertices visited on a graph $G$ during an optimally played game starting at vertex $v$.

Definitions

Let $G = (V, E)$ be a graph. 

- The distance between two vertices $u, v \in V$ is the length of the shortest path between them. 
- The eccentricity of a vertex $v \in V$ is the maximum distance between $v$ and any other vertex $u \in V$. 
- The radius of $G$ is the minimum eccentricity over $v \in V$. 
- The diameter of $G$ is the maximum eccentricity over $v \in V$. 
- A set of vertices $A \subset V$ is Director-closed if when the token is at any vertex in the set, regardless of the distance Explorer calls, Director can keep the token in the set. 
- A set of vertices $A \subset V$ is Explorer-friendly if no matter where the token is, Explorer has a strategy that will force the token into $A$ regardless of Director’s choices.

Trees

Let $G$ be a tree. For a vertex $v$, let $diamDist(v)$ be the shortest distance from $v$ to a longest path $P$ of $G$ (a diameter of $G$). Then 

$$f_d(G, v) = diamDist(v) + diam(G) + 1$$ 

$P$ is Director-closed, so Director’s strategy is to get to $P$ as soon as possible, while Explorer’s strategy is to stay away from $P$.

Example:

1. Explorer calls distance 1 to visit the vertex adjacent to $v$. 
2. Explorer calls 1 so Director must visit $b$ or $w$. If Director visits $w$, Explorer then calls 2 to force the token to $b$. 
3. Now Director can force the token to the diameter for any distance. 

All vertices along the diameter are visited like a path graph.

General Results

For any graph $G = (V, E)$ with vertex $v \in V$, 

- At the end of the game, the set of visited vertices always contains a Director-closed set. (not necessarily the entire set of visited vertices) 
- Lemma of Eccentricity: Let $A \subset V$ be Explorer-friendly. Then 
  $$f_d(G, v) \geq \min_{u \in A}ecc(u) + 1.$$ 
- $f_d(G, v) \geq radius(G) + 1 \geq \frac{diam(G)}{2} + 1$. 
- $f_d(G, v) \leq |B|$, where $B$ is any Director-closed subset of $V$ containing $v$. 

Paths

For a path graph $P_n$ with $n$ vertices and any starting vertex $v$, 

$$f_d(P_n, v) = n.$$ 

Non-Adaptive Explorer Strategies consist of a predetermined list of distances Explorer calls each round. 

For path graphs, non-adaptive Explorer strategies can always be used to visit all $n$ vertices in $n - 1$ rounds.

Characteristics of these strategies: 

- Explorer calls large distances. 
- For paths with an odd number of vertices, unless the token starts at the midpoint, it visits the midpoint last. 
- Each round, the token visits a previously unvisited vertex. 

Example: Below is the non-adaptive Explorer strategy for odd paths starting at a vertex that is not an endpoint or adjacent to an endpoint.

$$n - 1$$

$$V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow \ldots \rightarrow V_{n-1} \rightarrow V_{n-1}$$

1. Explorer calls $n - 1 - i$: to visit the furthest endpoint. 
2. Explorer calls $n - 1$: to visit the opposite endpoint. 
3. Explorer alternately calls $\frac{n}{2}$ and $\frac{n}{2}$ for $2i - 1$ rounds, stopping just before they revisit $v_i$. 
4. Explorer alternately calls $\frac{n}{2}$ and $\frac{n}{2}$ for the remaining $n - 2i - 2$ rounds to visit all remaining vertices.

Note: There is an adaptive Explorer strategy that will visit all vertices in $n - 1$ rounds that depends only on $n$, not $v$.

Lattices

Let $G$ be an $n$ by $n$ square lattice where $n$ is odd. Then 

$$f_d(G, v) = 2n - 1$$ 

for any $v$. 

Director can find a path $P$ of length $2n - 1$ through 

- any starting vertex 
- the center vertex 
- the corner closest to $v$, etc, and its opposite corner. 

$P$ is Director-closed 

$$f_d(G, v) \leq 2n - 1$$ 

Two opposite corners are Explorer-friendly since Explorer can always visit a corner $u_i$ by calling $ecc(v)$ for $v \in P$. Since $ecc(u_i) = 2n - 2$, then by the Lemma of Eccentricity, $f_d(G, v) \geq 2n - 2$. Thus, $f_d(G, v) = 2n - 1$.

When $n$ is even, $P$ does not exist since there is no center vertex. Each vertex has exactly one farthest vertex. 

Director can find a Director-closed set $A$ of $3n - 4$ vertices consisting of 

- two opposite sides $V_1$ 
- second row vertices non-adjacent to the opposite sides $V_1$ 

$$\Rightarrow f_d(G, v) \leq 3n - 4$$ 

for $v \in A$.

Future Questions

- What is $f_d(G, v)$ for other types of graphs? 
- For what other types of graphs do non-adaptive strategies allow Explorer to reach $f_d(G, v)$ vertices quickly? 
- How can we efficiently identify which subsets of a given graph are Director-closed and/or Explorer-friendly?

References


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