Bernoulli Numbers and Class Numbers of Cyclotomic Fields

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Bernoulli Numbers [2]:

History:

Jacob Bernoulli introduced the Bernoulli Numbers in 1713 in his book *Ars Conjectandi*. Recursively defined for $n > 2$ by

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

the numbers appear frequently in various areas of mathematics.

Generating Function:

The Bernoulli numbers have the exponential generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

Hence the nth Bernoulli number B_n is the coefficient of the nth term in the Taylor Series expansion of $\frac{t}{e^t-1}$ around 0.

Stirling Numbers:

The Stirling Numbers of the Second Kind $S(n,m)$ describe the number of ways to divide elements into non-empty subsets. This leads to the following definition of the Bernoulli numbers:

$$B_n = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m\, m!\, S(n,m)}{m+1}$$

Complex Analysis:

Bernoulli numbers appear in complex analysis in the integral

$$B_n = \frac{n!}{2\pi i} \oint \frac{z}{(e^z - 1)\, z^{n+1}} dz$$

This equality can be proved using the generating function.

Citations:

1. Su Hu, Min-Soo Kim, Pieter Moree, and Min Sha. "Irregular primes with respect to Genocchi numbers and Artin's primitive root conjecture". In:J. Number Theory 205 (2019), pp. 59–80. (page 3)
2. Tsuneo Arakawa, Tomoyoshi Ibukiyama, and Masanobu Kaneko.Bernoulli Numbers and Zeta Functions. Springer Monographs in Mathematics. With an appendix by Don Zagier. Springer, Tokyo, 2014, pp. xii+274 (pages 3-5)
3. Daniel A. Marcus.Number Fields. With a foreword by Barry Mazur. Springer, Cham,2018 (pages 10, 12, 13)

$$1, \frac{\pm 1}{2}, \frac{1}{6}, 0, \frac{-1}{30}, 0, \frac{1}{42}, 0, \frac{-1}{30}, 0, \frac{5}{66}$$
$$0, \frac{-691}{2730}, 0, \frac{7}{6}, 0, \frac{-3617}{510}, 0, \frac{43867}{798}, \ldots$$

Kummer's Theorem [1]:

An odd prime p is B- irregular if and only if p divides h_p^-.

What Does This Mean?

A prime p is B- irregular if p divides the numerator of at least one of the Bernoulli numbers $B_2, B_4, \ldots, B_{p-3}$. For any prime p, the relative class number h_p^- is the ratio of the class number of the pth cyclotomic field $\mathbb{Q}(\zeta_p)$ to the class number of the field $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$. Kummer's theorem states that p is B- irregular precisely when p divides h_p^-

Big Takeaway:

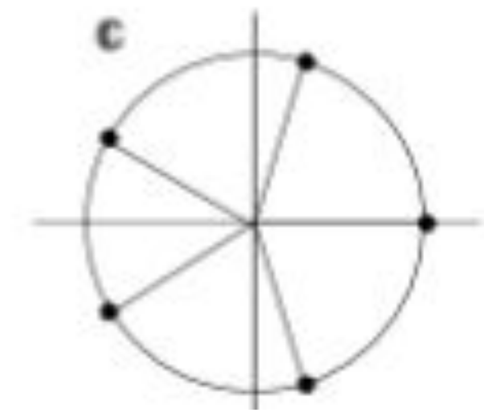
Kummer's theorem illustrates a deep connection between the Bernoulli numbers and algebraic number theory!

Cyclotomic Fields [3]:

The *nth cyclotomic field* is given by $\mathbb{Q}[\zeta_n]$ where ζ_n is a primitive nth root of unity. Consider the roots of the polynomial:

$$f(x) = x^n - 1$$

The complex numbers ζ that are the solutions to this polynomial are referred to as the *roots of unity*.



Fifth Roots of Unity

If $\zeta_n^{\,n}$ is the first power of ζ_n equal to 1 then ζ_n is a primitive root of unity. These field extensions are Galois and are in this tower of field extensions

Future Directions: The Class Number [3]

Loosely speaking, the order of the class group of a number field, the *class number*, is a measure of the degree to which unique factorization fails in the ring of integers of the field.

Because cyclotomic fields are Dedekind domains, unique factorization holds in the ring of integers of a cyclotomic field if and only if the field is a principal ideal domain. For example, the class group of $\mathbb{Z}$ (the ring of integers of $\mathbb{Q}$) is trivial because unique factorization holds in this ring. We are currently working to understand Kummer's Theorem in the context of $\mathbb{Q}(\zeta_{23})$, which is the lowest order cyclotomic field with nontrivial class group.

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