Modeling Cooperation Between Species

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Background

The Lotka-Volterra Model

The Lotka-Volterra (L-V) model is a classic approach to describe the population dynamics between predators and prey. As predator and prey species compete, their populations oscillate, creating a "boom and bust" cycle controlled by environmental limits.

$$\frac{dx}{dt} = \alpha x - \beta xy \qquad \frac{dy}{dt} = \mu xy - \delta y$$

Classic first-order differential L-V equations

- *x= prey species; y= predator species*
- *α = prey growth rate; β = impact of predation*
- *μ = predator growth due to predation; δ = natural death rate of predator*

While the competition described in the original equations can be made to represent cooperation by introducing a sign change, this fails to convey the complexity of a relationship that is simultaneously beneficial and deleterious. This type of interaction falls under the term of mutualism. In general, mutualistic relationships are those in which two species benefit from one another's presence.

Mutualism between Yucca Moths & Yucca Plants

The yucca plant-moth system is a prime example of an obligate mutualistic relationship (one species cannot survive without the other). The plants depend on moths for pollination, while the moth larvae rely on yucca plant fruit for sustenance after hatching. Holland and DeAngelis [1] propose a sufficiently detailed model describing the interaction between the plants and moths. Using [1] as a basis for our own work, we sought to establish a definitive model of mutualism, with applications in ecology and wider fields, such as mathematical finance.

Methods

We reduced the Holland-DeAngelis model to two dimensions and substituted heuristic descriptions with standard universal functions such as the logistic function, which have been extensively studied in the mathematical literature [2]. Additionally, we employed simulations of ordinary differential equations and discrete-time difference equations. The tools of choice were Mathematica and R, both of which are mathematical programming languages capable of solving dynamical systems numerically. In particular, we made use of their ability to do parameter searches as we explored the phase space of the mathematical models of species competition and cooperation. We also evaluated the stability of our systems using the Jacobian.

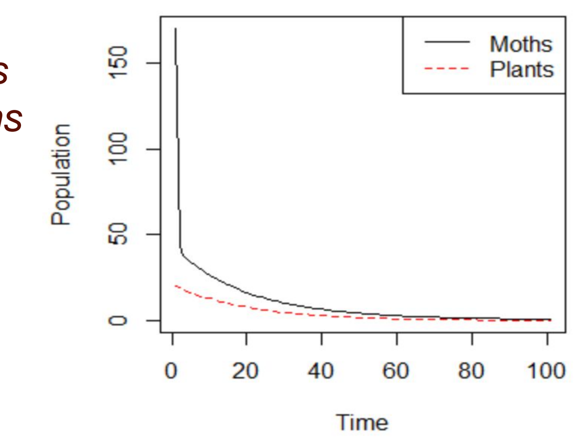
Results

As a first step, we did not distinguish between flowers and plants. We set flowers = plants, and focused only on the population of plants and moths.

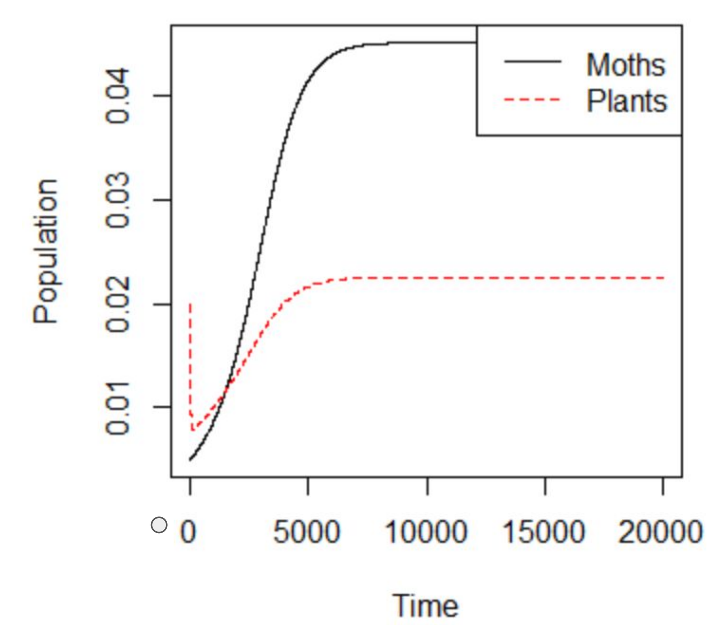
$$\frac{dp}{dt} = \alpha\delta\frac{m}{p}\left(1 - \delta\frac{m}{p}\right) - \mu p$$

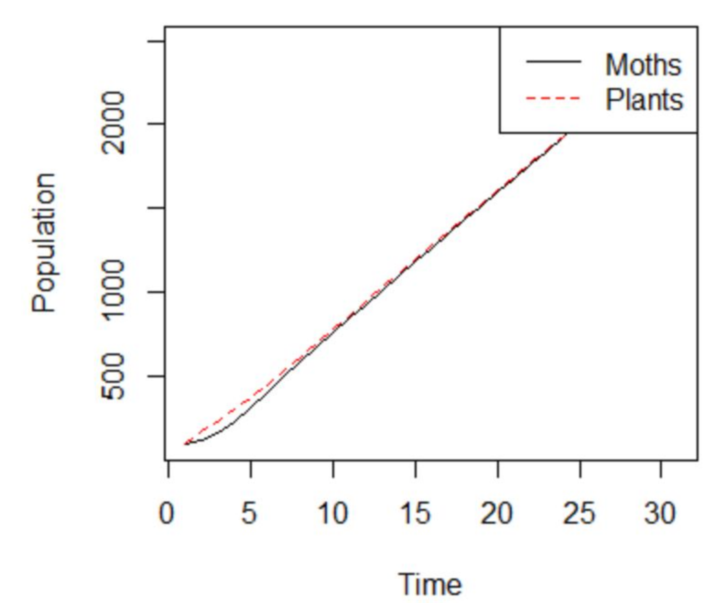
$$\frac{dm}{dt} = \beta pm - \mu\left(m^2\right)$$

- *p = # of plants*
- *m = # of moths*
- *α, δ, β, μ = arbitrary parameters*



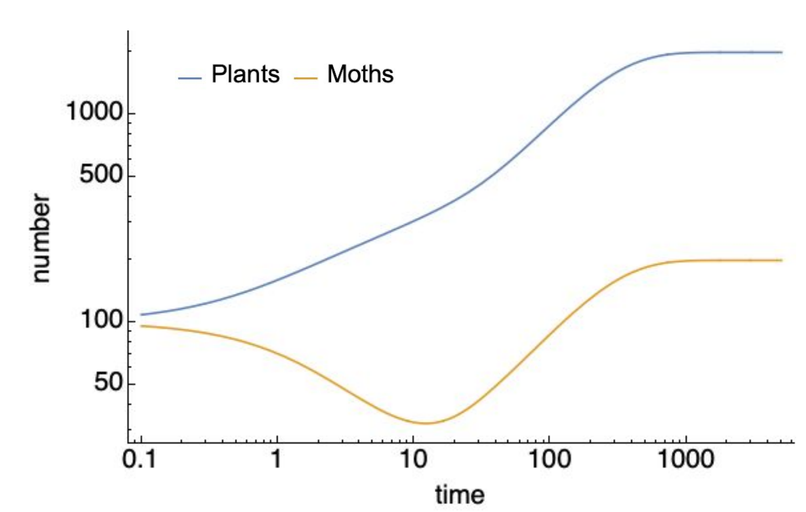
The above system resulted in populations that declined exponentially to approach near-zero equilibrium points. We also attempted to reduce the number of parameters (i.e. set αδ = γ), and remove the squared *m* in the moth population equation, but the steep decline of both populations remained. When we instead changed the initial population sizes (and later, parameters) for moths and plants, we were able to generate two interestingly stable systems.

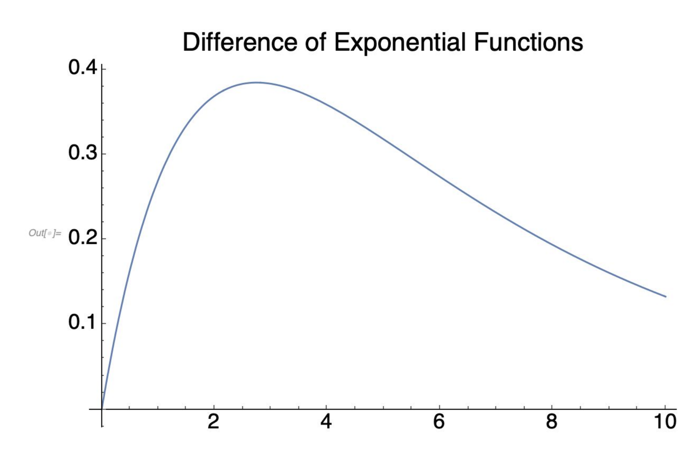




Following the format of the model in [1], we designed a model in which yucca plant fruit production was described by a difference of exponentials.

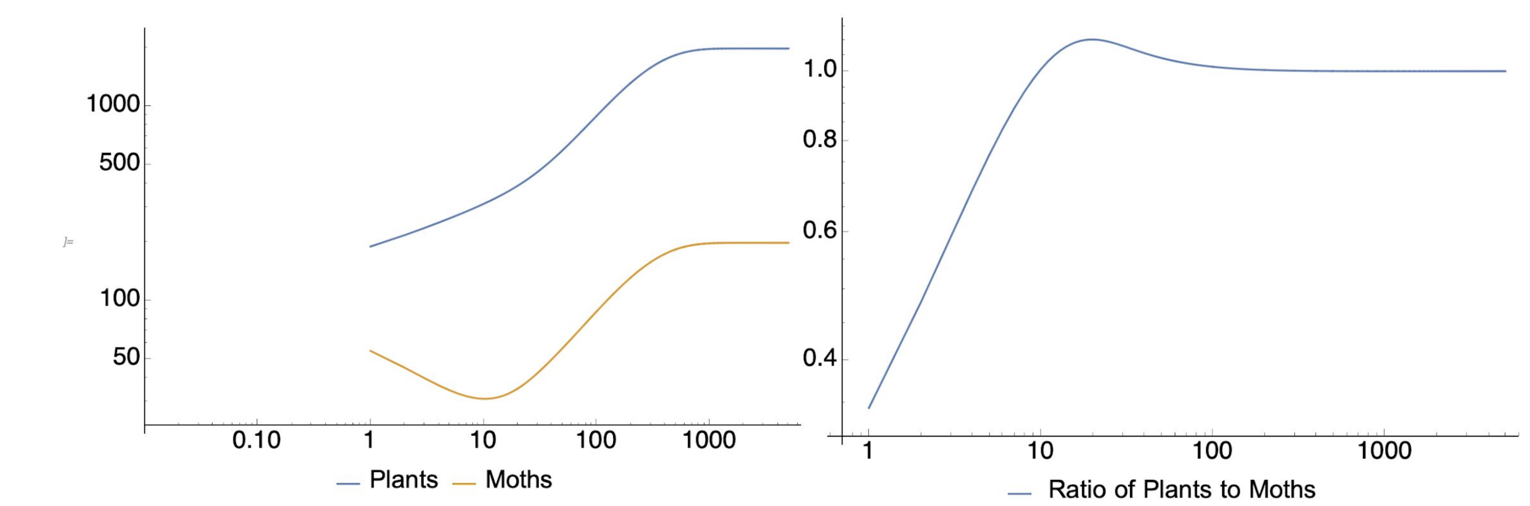
$$\frac{dp}{dt} = \alpha\left(e^{\left[-\varepsilon_1\frac{m}{p}\right]} - e^{\left[\varepsilon_2\frac{m}{p}\right]}\right) - \beta p \qquad \frac{dm}{dt} = \delta pm - \gamma m^2$$





Results continued

With further adjustments to the model, we found that the approach taken to reach stable populations could be highly usual. For example, as pictured in the graph below, moth and plant population sizes initially move *away* from equilibrium before reaching their final, stable values. The ratio of plants to moths is peculiar as well, even though it later settles towards a fixed value.



With our final model, we aimed to develop a system which more accurately represented the life cycles of yucca plants and moths. This model consists of three equations, each describing the dynamics of plant, moth, and fruit populations, respectively. An additional model also included an equation describing larvae population dynamics, so as to separate larvae and adult moths as in an age-structured population diagram. As the model equations are multiplicative, and hence nonlinear, analytical solutions were not possible, so we focused on numerical approaches for finding solutions.

$$\frac{dp}{dt} = \theta mf - \delta p \qquad \frac{dm}{dt} = \beta mf - \mu m^2 \qquad \frac{df}{dt} = \alpha p - \varphi f$$

Remaining Questions

In the future, we would like to further explore the stability of the various yucca plant-moth models conceived over the course of our investigation. For example, could it be proven that initializing a system with only positive parameters results in a system that stays positive? We also aim to answer the question of whether any of our models are relevant to other real-life examples of mutualism and cooperation, or solely the moths and plants at hand.

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[2] Strogatz, Steven.Â *Nonlinear Dynamics and Chaos: with Applications to Physics, Biology, Chemistry, and Engineering.* CRC Press, Taylor Et Francis Group, 1994.

Illustrations- Plant-- L. H. Bailey *Standard Cyclopedia of Horticulture* (New York, New York: The MacMillan Company, 1917); Moth -- Brian Hoffman courtesy of the Alberta Conservation Association and the Alberta Sustainable Resource Development.