Modeling Cooperation Between Species
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Background

The Lotka-Volterra Model

The Lotka-Volterra (L-V) model is a classic approach to describe the population dynamics between predators and prey. As predator and prey species compete, their populations oscillate, creating a "boom and bust" cycle controlled by environmental limits.

\[
\begin{align*}
\frac{dx}{dt} &= ax - bx y \\
\frac{dy}{dt} &= cyx - dy
\end{align*}
\]

Classic first-order differential L-V equations

While the competition described in the original equations can be made to represent cooperation by introducing a sign change, this fails to convey the complexity of a relationship that is simultaneously beneficial and deleterious. This type of interaction falls under the term of mutualism. In general, mutualistic relationships are those in which two species benefit from one another’s presence.

Mutualism between Yucca Moths & Yucca Plants

The yucca plant-moth system is a prime example of an obligate mutualistic relationship (one species cannot survive without the other). The plants depend on moths for pollination, while the moth larvae rely on yucca plant fruit for sustenance after hatching. Holland and DeAngelis [1] propose a sufficiently detailed model describing the interaction between the plants and moths. Using [1] as a basis for our own work, we sought to establish a definitive model of mutualism, with applications in ecology and wider fields, such as mathematical finance.

Methods

We reduced the Holland-DeAngelis model to two dimensions and substituted heuristic descriptions with standard universal functions such as the logistic function, which have been extensively studied in the mathematical literature [2]. Additionally, we employed simulations of ordinary differential equations and discrete-time difference equations. The tools of choice were Mathematica and R, both of which are mathematical programming languages capable of solving dynamical systems numerically. In particular, we made use of their ability to do parameter searches as we explored the phase space of the mathematical models of species competition and cooperation. We also evaluated the stability of our systems using the Jacobian.

Results

As a first step, we did not distinguish between flowers and plants. We set flowers = plants, and focused only on the population of plants and moths.

\[
\begin{align*}
\frac{dp}{dt} &= a\frac{m}{p} \left( 1 - \frac{m}{p} \right) - \mu p \\
\frac{dm}{dt} &= \beta \mu m - \mu (m^2)
\end{align*}
\]

The above system resulted in populations that declined exponentially to approach near-zero equilibrium points. We also attempted to reduce the number of parameters (i.e. set \(c = y\)), and remove the squared \(m\) in the moth population equation, but the steep decline of both populations remained. When we instead changed the initial population sizes (and later, parameters) for moths and plants, we were able to generate two interestingly stable systems.

Remaining Questions

In the future, we would like to further explore the stability of the various yucca plant-moth models conceived over the course of our investigation. For example, could it be proven that initializing a system with only positive parameters results in a system that stays positive? We also aim to answer the question of whether any of our models are relevant to other real-life examples of mutualism and cooperation, or solely the moths and plants at hand.

References:
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