Coprime Labeling of Ladder Graphs

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Abstract

A preprint by Ghorbani and Kamali provides an algorithm for the prime labeling of the ladder $P_n \times P_2$. In this poster, we describe our work to generalize these results to a coprime labeling of $P_n \times P_2$, where the set of labels consists of the $2n$ positive integers starting at k.

The Problem

Using an arbitrary interval of $2n$ consecutive integers, $\{k, k+1, \ldots k+2n-1\}$, when is it possible to coprime label $P_n \times P_2$?

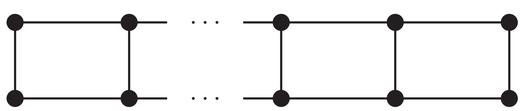
Introduction

The *greatest common divisor* or *GCD* of two integers a and b, denoted $\gcd(a,b)$ or (a,b), is the greatest positive integer that divides both a and b. In other words, we say that $(a,b) = d$ if $d|a$, $d|b$ and for all c such that $c|a$ and $c|b$, $c \leq d$.

Two integers a and b are said to be *coprime* if and only if $(a,b) = 1$.

A *graph* is simply a collection of vertices and edges.

A *ladder graph*, denoted $P_n \times P_2$, is a graph with $2n$ vertices that looks like



A *coprime labeling* of a graph is an assignment to a graph's vertices of the distinct positive integers from $\{k, \ldots, k+n\}$ in which adjacent vertices (those connected by an edge) are given coprime labels. That is, if two vertices labeled a and b are connected by an edge, then $(a,b) = 1$.

Example:

References

[1] Sara Kamali Ebrahim Ghorbani. Prime labeling of ladders. *Preprint at arXiv:1610.08849*, 2019.

Observations

1) Every vertex in a ladder is incident to at least two edges, so to complete a coprime labeling, every integer must be coprime with at least two other integers in the interval. A direct results of this is: *For all $m \in \mathbb{N}$ with $m \geq 2$, there exists an interval of length $2m$ such that $P_m \times P_2$ cannot be coprime labeled.*

Example: Consider $\{25, 26, 27, 28, 29, 30\}$. Notice that 30 is only coprime with 29, and the ladder graph requires each integer to be coprime with at least two other integers in the interval. Thus, this interval cannot be used to coprime label $P_3 \times P_2$.

More specifically, every integer must be coprime with at least two other integers *of opposite parity* since even integers cannot be placed adjacent to each other.

2) There cannot exist more than two integers that are coprime with exactly two other integers in the interval.

This is concerned with using the corners of the ladder for integers that are highly divisible. Let m, k be the two integers that are coprime with exactly two other integers. That is, m is coprime with only $m+1$ and $m-1$ and k is coprime with only $k-1$ and $k+1$. Then m and k must be even, otherwise they would be coprime with $m \pm 2$ and $k \pm 2$ respectively. Thus, they need to be placed on the opposite ends of the ladder. The corners are now filled and any other integer in the interval has to be placed inside the ladder.

3) For any composite integer m in the interval that cannot be placed in the corners, there must exist an integer of the form $m+p$ (or $m-p$) in the interval where $p > 2$ is the smallest prime that does not divide m, so that m can be placed adjacent to $m+1$,$m-1$ and $m+p$ (or $m-p$).

Ongoing Research

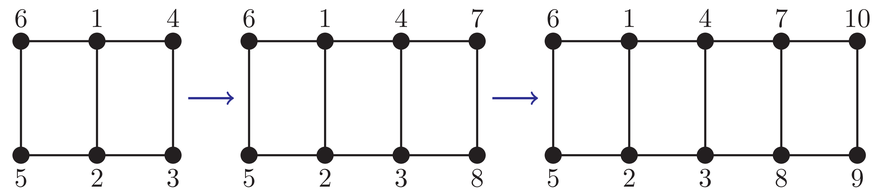
- Analyzing different cases to verify that the construction technique works for any interval that satisfies the conditions above.
- Can an interval that cannot be used to coprime label a ladder graph be extended so that the resulting interval can be used for a coprime labeling? If so, how many numbers need to be added?

Coprime labeling $P_n \times P_2$ using $\{1, \ldots, 2n\}$ (Ghorbani, Kamali)

1. Pair up consecutive integers of the form
   - $6k$ and $6k-1$
   - $6k-2$ and $6k-3$
   - $6k-4$ and $6k-5$

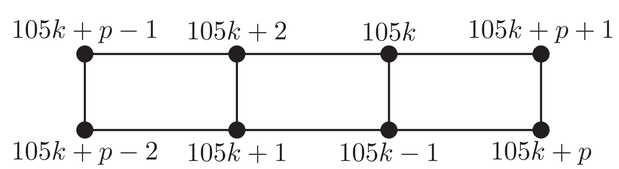
   to label the vertices connected by a vertical edge. Since two consecutive integers are always coprime, this allows us to make sure the GCD condition is satisfied on the vertical edges.
2. Build the ladder adding one pair at a time, making sure the GCD condition is satisfied at every step.
3. Recall that $(a,b) = (a, b-a)$. The integers 3, 5 and 7 are used as values for $b-a$, as using bigger numbers introduces more primes and more GCD cases.

Example:



General construction

- Using Mathematica, we verified that with this technique, a coprime ladder can be built using an interval starting at 1 and ending at 105. Because the differences between the labels are $3, 5$ and 7, by The Chinese Remainder Theorem, the GCD patterns repeat every $3*5*7 = 105$ integers. Thus, we get GCD condition failures precisely at numbers of the form $105k$.
- By observation (3), in order to be able to coprime label the arbitrary interval containing an integer of the form $105k$, there must exist an integer of the form $105k+p$ (or $105k-p$) in the interval where $p > 2$ is the smallest prime that does not divide $105k$. Depending on the parity of $105k+p$, we can make possible swaps in the ladder. For example, if $105k+p$ is even, the following construction



  is one possible switch that can be made to the ladder built using Ghorbani and Kamali's technique.
- More generally, if p is the smallest prime greater than 2 such that $p \nmid 105k$, then $(105k+m, n) = 1$ whenever $|n| < p$ and $|m| < 11$.

  By our choice of p, we are able to use this lemma to prove that making the switch gives a coprime labeling. We address six different cases depending on the value of $105k+p$ under modulo 6 and make similar swaps to get a coprime labeling .