Surfaces with Braided Boundaries in Blow-ups of $D^2 \times D^2$

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Supported by NSF grant DMS-1839968

1. Goal

Monodromy is a way to measure something as we travel around a loop. The chart description developed by Kamada links monodromy representation to topological surfaces. Our goal is to use the framework Kamada developed to explore surfaces with braided boundaries in blow-ups of $D^2 \times D^2$ which has potential applications to sympletic geometry.

2. Preliminaries

2.1 Closed, Pure Braids

Definition:
A closed, pure braid is a braid where:

- (Pure) Each strand starts and ends in the same position
- (Closed) The ends are paired so that each strand is a circle.

2.2 Blow-ups

Idea:
A blow-up is a way to resolve intersections by 'gluing-in' a projective space. For example, blow-ups in $\mathbb{C}^2$ locally look like

$$E = \{(\ell, v) \in \mathbb{C}P^1 \times \mathbb{C}^2 \mid \ell \text{ is a line in } \mathbb{C}^2, \text{ and } v \text{ is a vector in } \ell\}.$$

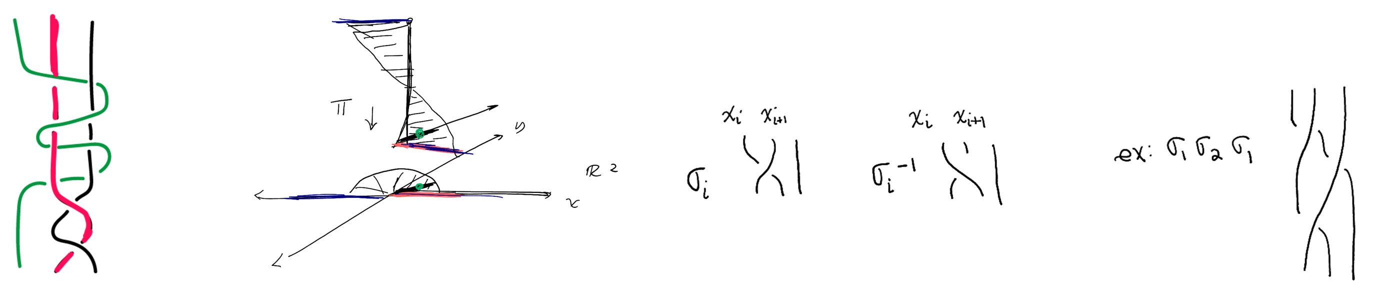


Figure 1: *Pure braid example (left), Blow-up in $\mathbb{R}^2$ (center), Braid group gen. (right)*

2.3 Research Problem

We are looking at surfaces in the blowup of $D^2 \times D^2$. In particular, we are looking at the surfaces with closed, pure braids β in the solid torus ($S^1 \times D^2$) as their boundaries. Let β be a pure, closed braid in the solid torus $S^1 \times D^2$, having n strands. For an integer $\ell \geq 0$, let X_ℓ denote a blowup of $D^2 \times D^2$ at ℓ distinct points.

Definition: Spanning surface
A *spanning surface*, S, with β as the boundary braid in X_ℓ is a properly embedded surface with the following properties:

1. S is diffeomorphic to a disjoint union $S = S_1 \cup \cdots \cup S_n$ with each S_i diffeomorphic to D^2.
2. For each $j = 1, \ldots, n$ the restriction of $\pi : X_\ell \to D^2$ gives a diffeomorphism from S_j to D^2.
3. The above implies ∂S is contained in $\partial D^2 \times D^2$. This boundary is the closed braid, β

We say the spanning surface is *essential* if for each exceptional sphere E_j ($j = 1, \ldots, \ell$), $S \cap E_j \neq \emptyset$.

Given a boundary braid β and the number of blowups, we want to determine how many essential spanning surfaces there are for β in X_ℓ up to isotopy.

3. Our Work

3.1 Charts

Definition: Chart description

- A graph in the disk with two kinds of vertices (black and white)
- with edges that are oriented and labeled by and integer (braid group gen.)
- and there are rules about what collections of edges can come into a given vertex.

Intuition: A spanning surface comes with a projection map: $\pi : D^2 \times D^2 \supset S \to D^2$. The chart is contained in the target of this map and the various labels help you to 'reconstruct' the spanning surface over 'cross-sections' of the disk.

We have three main ways of filling in our chart, each of which corresponds to a braid isotopy:

- Reidemeister Type II moves: $\sigma_j(\sigma_i^{-1}\sigma_i)\sigma_k = \sigma_j\sigma_k$
- Reidemeister Type III moves (i.e. the braid relation): $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$
- Far Commutativity: $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$

Definition: White vertex
A *white vertex* represents a place in the chart where a relation or far commutativity takes place.

Definition: Black vertex
We encode a blow-up of $D^2 \times D^2$ via a *black vertex* in our chart.

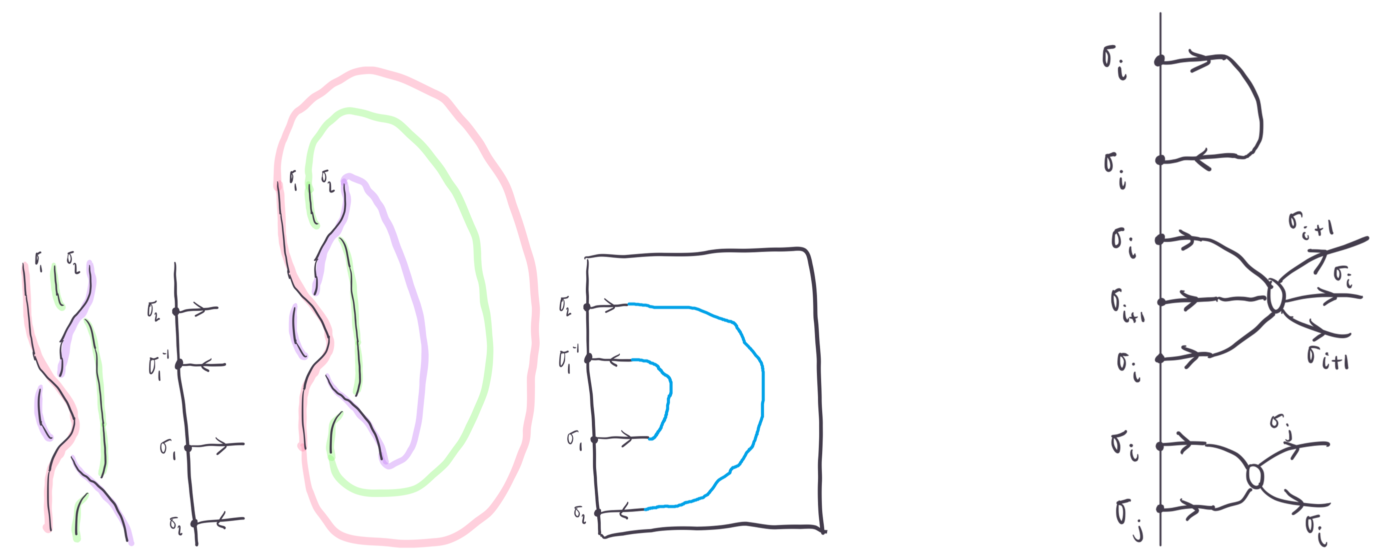


Figure 2: *Closing up a braid/chart (left), Chart move examples (right)*

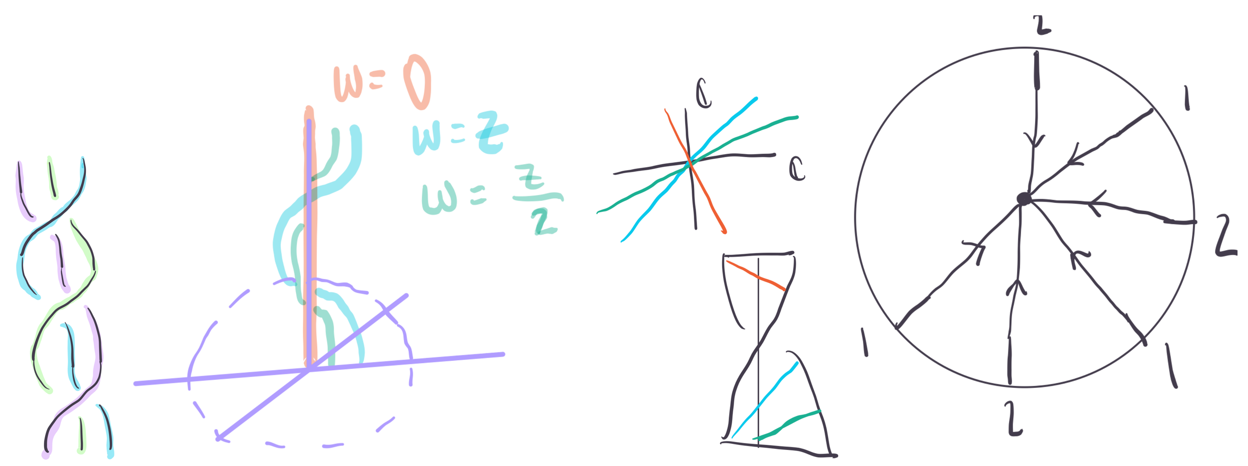


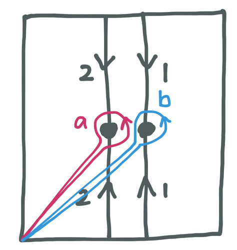
Figure 3: *Black vertices and full-twists*

3.2 Monodromy Representation

In our setting, the monodromy representation is a homomorphism:

$$\rho : \underbrace{\pi_1(D^2 \setminus \{\text{black vertices}\})}_{\text{based loops in a chart}} \to \mathrm{Br}_n \text{ where}$$

each generator of $\pi_1(D^2 \setminus \{\text{black vertices}\})$ maps to some conjugate of a full twist. More concretely, we can see what the homomorphism does by 'reading-off letters.'



Example:

$$a \mapsto 22$$
$$b \mapsto 2112^{-1}$$

4. Theorem in Progress

4.1 The Monodromy Principle

Spanning surface $\longleftrightarrow$ Monodromy representation $\longleftrightarrow$ Chart description.

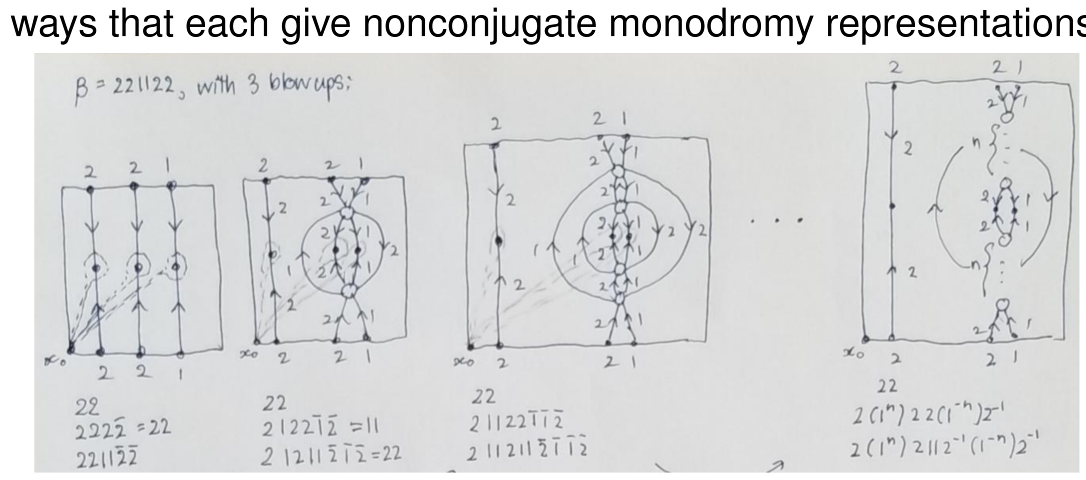
In general, Kamada has proved that "Monodromy representation $\longleftrightarrow$ Chart description"; we extended this idea by using black vertices to represent blow-ups. Further, we proved that any spanning surface can be described by a chart, proving the left-to-right direction of this principle in the context of blow-ups.

Theorem in progress:
In the setting of surfaces in blowups of $D^2 \times D^2$, there is a correspondence between spanning surfaces and monodromy representation.

Consequence of Theorem in progress:
Spanning surfaces are isotopic (relative to the boundary) if their charts determine the same monodromy or conjugate monodromies ($\rho \sim g\rho g^{-1}$ for some $g \in \mathrm{Br}_n$).

5. Future Directions

Using the language of chart descriptions, we can also ask: Is it possible to fill in a chart in two different ways that each give nonconjugate monodromy representations? Yes!



Question:
Is it possible for some braid, β, to have infinitely many spanning surfaces?

References

[1] Kamada, Seiichi (J-HROSE-M2) *Graphic descriptions of monodromy representations.* (English summary) *Topology Appl.* 154 (2007), no. 7, 1430-1446. 57M07

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