ON MINIMALITY OF TWO-BRIDGE KNOTS OF THE FORM $\mathfrak{b}(7k\pm3,7)$

Anh Tran, Thi Phuong Anh Pham, Riley Stephens, James Boyer

The University of Texas at Dallas

Abstract

A two-bridge knot, or rational knot, has the form $\mathfrak{b}(p,q)$ such that $\frac{p}{q} = k + \frac{1}{a+\frac{1}{b}}$. Another notation for $\mathfrak{b}(p,q)$ is then $[k,a,b]$. This type of knot gives k-long, a-long, and b-long twists. While the minimality of rational knots for any q up to 5 is known, we want to study the minimality of knots with $q=7$ of the form $\mathfrak{b}(7k\pm 3, 7)$ (because the cases where $p = 7k\pm1$ or $p = 7k\pm 2$ have been studied).

Background

Definition: Chebyshev Polynomials

- $\forall t \in \mathbb{C}$ let $S_0(t) = 1$
- Let $S_1(t) = t$
- Then $S_n(t) = tS_{n-1}(t) - S_{n-2}(t) \forall n \in \mathbb{Z}$
- Furthermore, $S_n^2(t) + S_{n-1}^2(t) - tS_n(t)S_{n-1}(t) = 1$

Structure-Preserving Forms

The ***fundamental group*** of the complement of a knot (usually just called the fundamental group) is the group of all "loops" (continuous closed paths) through a fixed point in the ambient space, with equivalence up to homotopy. Equivalence can be demonstrated with Reidemeister moves. As fundamental groups are invariant for knots up to homotopy, they preserve minimality.

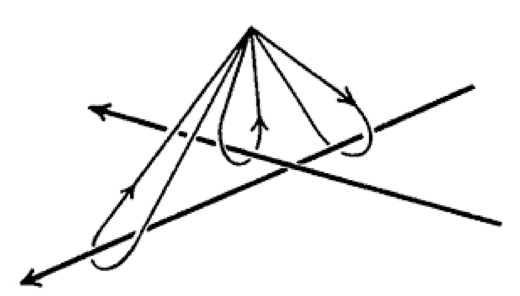

Figure 1: Loops associated to the three arcs in this knot crossing. From [1].

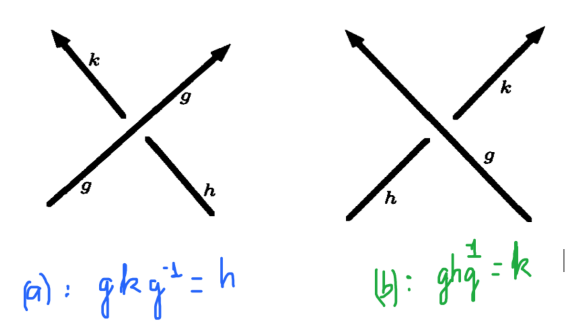

Figure 2: Labelling knot diagrams. From [2].

Definition: Wirtinger Presentation

- Let K be a knot in 3-space with a known fundamental group
- Then K has a *Wirtinger presentation* of the form
$$\langle g_1, g_2, ..., g_k | w g_i w^{-1} = g_j\rangle$$
- w is a word in finite generators $\{g_1, g_2, ..., g_k\}$, the unique loops for K

For a Wirtinger presentation in two generators (say a and b), the presentation can be expressed as a non-Abelian *representation* of the form

$$\langle A, B \in SL_2 \subset \mathbb{C} | AW = WB\rangle$$

where A and b correspond to a and b respectively, and the word W has the form of w, substituting a, b, and multiplication with A, B, and matrix multiplication.

Background Cont.

Minimality

A knot is called ***minimal*** if its knot group admits epimorphisms onto the knot groups of only the trivial knots and itself. The $SL_2[\mathbb{C}]$ *character* of a fundamental group's representation is the trace of the representation. The $SL_2[\mathbb{C}]$ *character variety* is a space of equivalence classes of representations by character. The ***Riley polynomial***, $R(x,z)$, which appears as a factor in the anti-diagonal entries of $WB - AW$, is a polynomial in $x \in \mathbb{Z}$ with coefficients in $\mathbb{C}[z]$ given by $z = tr(BA)$. Now,

$$AW = WB \implies AW - WB = O$$

which amounts to saying $R(x,z) = 0$. If the Riley polynomial is irreducible, then the character variety has only a single component. If the non-abelian character variety is irreducible, then the knot is minimal, as demonstrated by Nagasato et al in [3] .

Research Problem

We first study the presentation of the knot group. Next, we find its representation. Then, we examine the non-abelian character variety of the group, which amounts to studying the irreducibility of the Riley polynomial.

Our work

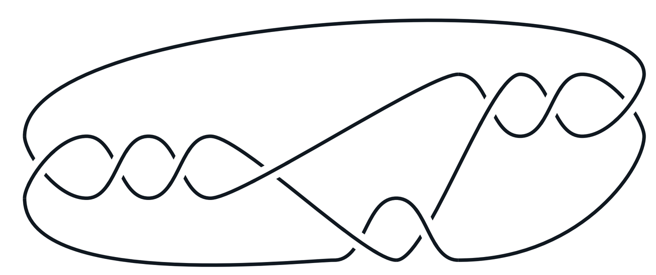

Figure 3: The simplest $\mathfrak{b}(7(2n)+3,7)$ knot, [4,2,3].

Presentation of the knot group

Let K denote the two-bridge knot $\mathfrak{b}(7(2n)+3,7)$. The knot group of K admits the following presentation:

$$\pi_1(K) = \langle a, b | aw = wb\rangle$$

where

$$w = (ba)^n a (ba)^{-n} b (ba)^n b (ba)^{-n} b^{-1} (ba)^n a (ba)^{-n} b (ba)^n$$

Representation of the knot group

Suppose $\rho : \pi_1(K) \to SL_2(\mathbb{C})$. By Riley's results [4], up to conjugation we can assume:

$$\rho(a) = \begin{pmatrix} s & 1 \\ 0 & s^{-1}\end{pmatrix} \text{ and } \rho(b) = \begin{pmatrix} s & 0 \\ z - s^2 - s^{-2} & s^{-1}\end{pmatrix}$$

where $z = \text{tr}\rho(ba)$ and $(s,z) \in \mathbb{C}^2$ such that $\rho(wb) - \rho(aw) = 0$

We calculate

$$\rho((ba)^n) = \begin{pmatrix} S_n(z) - s^{-2}S_{n-1}(z) & s^{-1}S_{n-1}(z) \\ s^{-1}(z - s^2 - s^{-2})S_{n-1}(z) & S_n(z) - (z - s^{-2})S_{n-1}(z)\end{pmatrix}$$

Let $x = \text{tr}\rho(a) = \text{tr}\rho(b) = s + s^{-1}, X = S_n(z), Y = S_{n-1}(z)$.

$$\rho(wb) - \rho(aw) = \begin{pmatrix} 0 & R(x,z) \\ -(z - s^2 - s^{-2})R(x,z) & 0\end{pmatrix}$$

where $R(x,z) = \alpha x^6 + \beta x^4 + \gamma x^2 + \delta$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}[z]$ We can rewrite $R(x,z) = \alpha t^3 + \beta t^2 + \gamma t + \delta$.

Our work Cont.

Studying irreducibility of the Riley polynomial

We first calculate the discriminant of $R(x,z)$:

$$D_1 = (X-Y)^6 Y^6 (z-2)^3 A$$

where A is an expression in X, Y, z.

Next, we assume that $R(x,z)$ is reducible, thus has a linear factor in its factorization. We calculate the discriminant of that factorization:

$$D_2 = (X-Y)^6 Y^6 (z-2)^3 B$$

where B has a repeated factor.

We conjecture that when gcd$(\alpha, \beta, \gamma, \delta) = 1$, then $R(x,z)$ is irreducible. Thus, we need to prove that A has no repeated factor.

Direction

We assume that the Riley polynomial is reducible (when gcd$(\alpha, \beta, \gamma, \delta) = 1$). We try to reach a contradiction using the following directions:

- Exploring the degree of the coefficients
- Using the method of depressing the cubic

Theorem in progress

For any integer $n \geq 0$, the following holds:

- The nonabelian character variety of the two-bridge knot $\mathfrak{b}(7(2n)+3,7)$ is reducible iff $n \equiv 0 \pmod 3$

Remarks

In the case of the knot $\mathfrak{b}(7(2n)-3,7)$, we conjecture that the result is the same when gcd$(\alpha, \beta, \gamma, \delta = 1)$; i.e, the Riley polynomial is irreducible in that case. Thus, the proof will be similar to that of the knot $\mathfrak{b}(7(2n)+3,7)$

Acknowledgements

We would like to thank Dr. Anh Tran for guiding us through the project and James Boyer for his contribution to the project.

References

[1] Larsen Linov. *An introduction to Knot Theory and the Knot Group.* URL: http://math.uchicago.edu/~may/REU2014/REUPapers/Linov.pdf.

[2] Charles Livingston. *Knot Theory.* American Mathematical Society, 1996. URL: http://ebookcentral.proquest.com/lib/utd/detail.action?docID=3330352.

[3] Fumikazu Nagasato, Masaaki Suzuki, and Anh T. Tran. "On minimality of two-bridge knots". In: *International Journal of Mathematics* 28.03 (2017).

[4] Robert Riley. "Nonabelian representations of 2-bridge knot groups". In: *Quart. J. Math* 35.2 (1984).