A New Frobeneus Template in a Matrix Ring

Timothy Eller, Jakub Kraus, Yuki Takahashi, Zhichun (Joy) Zhang

Auburn University REU 2020 with professors Peter Johnson and Overtoun Jenda

Classical Frobenius Problem

The general goal of this research was to generalize the classical Frobeius problem, which is in $\mathbb{N}$, to different rings. To do this, we first introduce the classical Frobenius problem.
Let $MN(a_1,\dots,a_n) = \{\lambda_1 a_1 + \cdots + \lambda_n a_n : \lambda_1,\dots,\lambda_n \in \mathbb{N}\}$. Note that $MN(a_1,\dots,a_n) \subseteq \mathbb{N}$.

Sylvester and Frobenius' Theorem

If $a_1,\dots,a_n \in \mathbb{Z}^+$ and $\gcd(a_1,\dots,a_n) = 1$, then for some $w \in \mathbb{N}$, $w + \mathbb{N} \subseteq MN(a_1,\dots,a_n)$.

Example

If $a_1 = 4, a_2 = 5 \in \mathbb{Z}^+$, then we have
$MN(4,5) = \{0,4,5,8,9,10,12,13,14,15,\dots\}$
and $12 + \mathbb{N} \subseteq MN(4,5)$.
When $n = 2$, the smallest such w, denoted $\chi(a_1,a_2)$, is $(a_1-1)(a_2-1)$.

Frob$(a_1,\dots,a_n)$

For $a_1,\dots,a_n \in \mathbb{Z}^+$, $\text{Frob}(a_1,\dots,a_n) = \{w \in \mathbb{N} : w + \mathbb{N} \subseteq MN(a_1,\dots,a_n)\}$

Example

$\text{Frob}(4,5) = \{12,13,14,15,\dots\} = 12 + \mathbb{N}$.

Generalizing Frobenius Problem

In the classical case, the ring is $\mathbb{Z}$ and

1. coprime sequence is from $\mathbb{Z}^+$
2. coefficients of the linear combinations are from $\mathbb{N}$
3. $MN(a_1,\dots,a_n) \subseteq \mathbb{N}$

In order to generalize the Frobenius problem from $\mathbb{N}$ to other rings, these three roles that $\mathbb{Z}^+$ and $\mathbb{N}$ play in the classical problem have to be replaced with appropriate subsets of respective rings, using a Frobenius template.

Frobenius Template

A *Frobenius template* in a ring R is a triple (A, C, U) such that

1. A is a nonempty subset of R
2. C and U are additive monoids in R
3. for each $\alpha_1,\dots,\alpha_n \in A$, $MN(\alpha_1,\dots,\alpha_n) = \{\sum_{i=1}^n \lambda_i\alpha_i : \lambda_1,\dots,\lambda_n \in C\} \subseteq U$.

In the classical case, the template is $(\mathbb{Z}^+, \mathbb{N}, \mathbb{N})$.

Frobenius Set

Given the choice of template, the *Frobenius set* of a list $\alpha_1,\dots,\alpha_n \in A$ is $\text{Frob}(\alpha_1,\dots,\alpha_n) = \{w \in R : w + U \subseteq MN(\alpha_1,\dots,\alpha_n)\}$. Given (A, C, U), we have two aims:

1. Determine for which lists $\alpha_1,\dots,\alpha_n \in A$ is it true that $\text{Frob}(\alpha_1,\dots,\alpha_n) \neq \emptyset$.
2. For each list $\alpha_1,\dots,\alpha_n$ with $\text{Frob}(\alpha_1,\dots,\alpha_n) \neq \emptyset$, describe $\text{Frob}(\alpha_1,\dots,\alpha_n)$.

In the classical case, for $a_1,\dots,a_n \in \mathbb{Z}^+$,
$\text{Frob}(a_1,\dots,a_n) \neq \emptyset \iff \gcd(a_1,\dots,a_n) = 1$
$\text{Frob}(a_1,\dots,a_n) \neq \emptyset \implies \text{Frob}(a_1,\dots,a_n) = \chi(a_1,\dots,a_n) + \mathbb{N}$.

Prior Results

Here are some examples of previously studied Frobenius templates.

1. $(\mathbb{Z}^+, \mathbb{N}, \mathbb{N})$ in the ring $\mathbb{Z}$
2. $(\mathbb{N}[\sqrt{m}]\backslash\{0\}, \mathbb{N}[\sqrt{m}], \mathbb{N}[\sqrt{m}])$ in the ring $\mathbb{Z}[\sqrt{m}]$
3. $(\mathbb{Z}[\sqrt{m}]\cap(0,\infty), \mathbb{Z}[\sqrt{m}]\cap(0,\infty), \mathbb{Z}[\sqrt{m}]\cap(0,\infty))$ in the ring $\mathbb{Z}[\sqrt{m}]$
4. $(\mathbb{N}[i]\backslash\{0\}, \mathbb{N}[i], U(\alpha_1,\dots,\alpha_n) = V[\min_j \theta(\alpha_j), \frac{\pi}{2} + \max_j \theta(\alpha_j)])$ in the ring $\mathbb{Z}[i]$

Note that $S[a] = \{s_1 + s_2 a : s_1, s_2 \in S\}$.

Set-up of a Matrix Ring

- Q is a ring that contains the multiplicative identity 1
- $R = Q \times Q$
- R contains the multiplicative identity $(1, 0)$.
- $1 \in \sum_{i=1}^n a_i c_i$
- There is an isomorphism between $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ and (a, b)
- $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \times \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} = \begin{bmatrix} ac & ad+bc \\ 0 & ac \end{bmatrix}$
- The multiplication in R is defined by $(a,b)\cdot(c,d) = (ac, ad+bc)$.
- R is commutative because $(a,b)\cdot(c,d) = (ac, ad+bc) = (ca, cb+da) = (c,d)\cdot(a,b)$.

The New Frobeneus Template

- Q is a subfield of $\mathbb{R}$
- $Q = \mathbb{Z}$

The Template: $(A(Q), C(Q), U(Q)) = \left((Q\cap(0,\infty)) \times (Q\cap[0,\infty)), (Q\cap[0,\infty))^2, (Q\cap[0,\infty))^2\right)$

Our Results

Theorem 1 (a subring of $\mathbb{R}$):

Let Q be a subring of $\mathbb{R}$ containing 1. For any positive integer n, let $(\alpha_1,\dots,\alpha_n)$ be a sequence in $A(Q) \setminus (\mathbb{R} \times \{0\})$. Then $\text{Frob}(\alpha_1,\dots,\alpha_n) = \emptyset$.

Theorem 2 (a subfield of $\mathbb{R}$):

Let Q be a subfield of $\mathbb{R}$. For any positive integer n, let $(\alpha_1,\dots,\alpha_n)$ be a sequence of 2-tuples $\alpha_i = (a_i, b_i) \in A(Q)$ (for $i = 1,\dots,n$) satisfying $b_1 = 0$. Then $\text{Frob}(\alpha_1,\dots,\alpha_n) = MN(\alpha_1,\dots,\alpha_n) = U(Q) = A \cup \{0\}$.

Example

For a sequence of three 2-tuples $\alpha_1 = (3,0)$, $\alpha_2 = (3.1, 3.14)$, $\alpha_3 = (3.141, \pi)$,
$\text{Frob}(\alpha_1,\alpha_2,\alpha_3) = MN(\alpha_1,\alpha_2,\alpha_3) = U(Q) = (\mathbb{R}^+ \cup \{0\})^2$

Theorem 3 ($\mathbb{Z}$)

For $n \geq 2$, let $(\alpha_1,\dots,\alpha_n)$ be a sequence of 2-tuples $\alpha_i = (a_i, b_i) \in \mathbb{Z}^+ \times \mathbb{N}$ (for $i = 1,\dots,n$) satisfying $\gcd(a_1,\dots,a_n) = 1$ and $b_1 = 0$. Then $\left(\chi(a_1,\dots,a_n), \chi(a_1,\dots,a_n) + (a_1-1)\left(\sum_{i=2}^n b_i\right)\right) + \mathbb{N}^2 \subseteq \text{Frob}(\alpha_1,\dots,\alpha_n)$.

Corollary ($\mathbb{Z}$)

For $n \geq 2$, let $(\alpha_1,\dots,\alpha_n)$ be a sequence of 2-tuples $\alpha_i = (a_i, b_i) \in Z^+ \times \mathbb{N}$ (for $i = 1,\dots,n$) satisfying $\gcd(a_1,\dots,a_n) = 1$. $\text{Frob}(\alpha_1,\dots,\alpha_n)$ is nonempty if and only if at least one $b_i = 0$.

Theorem 4 ($\mathbb{Z}$)

Suppose that $a_1, a_2 \in \mathbb{Z}^+, b \in \mathbb{N}$, a_1 and a_2 are coprime in $\mathbb{Z}$, and $\alpha_1 = (a_1, 0), \alpha_2 = (a_2, b)$. Then $\text{Frob}(\alpha_1,\alpha_2) = (\chi(a_1,a_2), \chi(a_1,a_2) + b(a_1-1)) + \mathbb{N}^2$

Example

For a sequence of two 2-tuples $\alpha_1 = (2,0)$ and $\alpha_2 = (3,4)$, $\text{Frob}(\alpha_1,\alpha_2) = ((2-1)\cdot(3-1), (2-1)\cdot(3-1) + 4\cdot(2-1)) + \mathbb{N}^2 = (2,6) + \mathbb{N}^2$.

$\mathbb{Z}$, $n > 2$?

Counter-example Let $\alpha_1 = (3,0)$, $\alpha_2 = (5,2)$, and $\alpha_3 = (7,4)$. Then it can be shown that $(5,16) \in \text{Frob}(\alpha_1,\alpha_2,\alpha_3) + \mathbb{N}^2$, but $(5,16) \notin \{\chi(a_1,a_2,a_3), \chi(a_1,a_2,a_3) + (a_1-1)(b_2+b_3)\} + \mathbb{N}^2$. In fact, it can be shown that $\text{Frob}(\alpha_1,\alpha_2,\alpha_3) = (5,9) + \mathbb{N}^2$.