Rainbow Cells:
Tracking Trends in Cell Division Using Motzkin Paths

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Background

Let P be a single proliferated cell. Suppose P has undergone n cell divisions. For each $i = 1, 2, 3, \ldots, n$ let C_i be the resulting cluster of cells after the ith division. Let k_i be the number of proliferated cells in cluster C_i. Since we need the existence of a proliferated cell for the division to occur, then P must stop dividing if there exists i such that $k_i = 0$. We want to track the divisions needed to create cluster C_i. If P divides into two proliferated cells then $k_{i+1} = k_i + 1$. If P divides into two differentiated cells then $k_{i+1} = k_i - 1$. If P divides into one proliferated cell and one differentiated cell then $k_{i+1} = k_i$. Let a, b, and c be the likelihood $k_{i+1} = k_i + 1$, $k_{i+1} = k_i$, and $k_{i+1} = k_i - 1$ respectively.
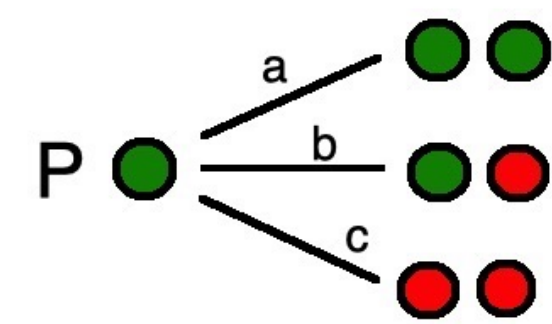

Figure 1: Depiction of Proliferated Cell Division

Suppose we wanted to map these divisions on the xy-coordinate system. Let x represent the number of divisions and y represent the number of proliferated cells at said division. Consider division i for proliferated cell P. If $k_{i+1} = k_i + 1$ then this is considered an up step on the xy-plane. If $k_{i+1} = k_i$ then this is considered a level step on the xy-plane. If $k_{i+1} = k_i - 1$ then this is considered a down step on the xy-plane. A Motzkin path is a random walk that starts at the origin and ends on the x-axis using three different steps; an up step $(1, 1)$, a down step $(1, -1)$, or a level step $(1, 0)$. For this reason, we can use Motzkin paths to model cell proliferation. The number of Motzkin paths that start at $(0,0)$ and end at the point $(n, 0)$ are found in the following sequence:

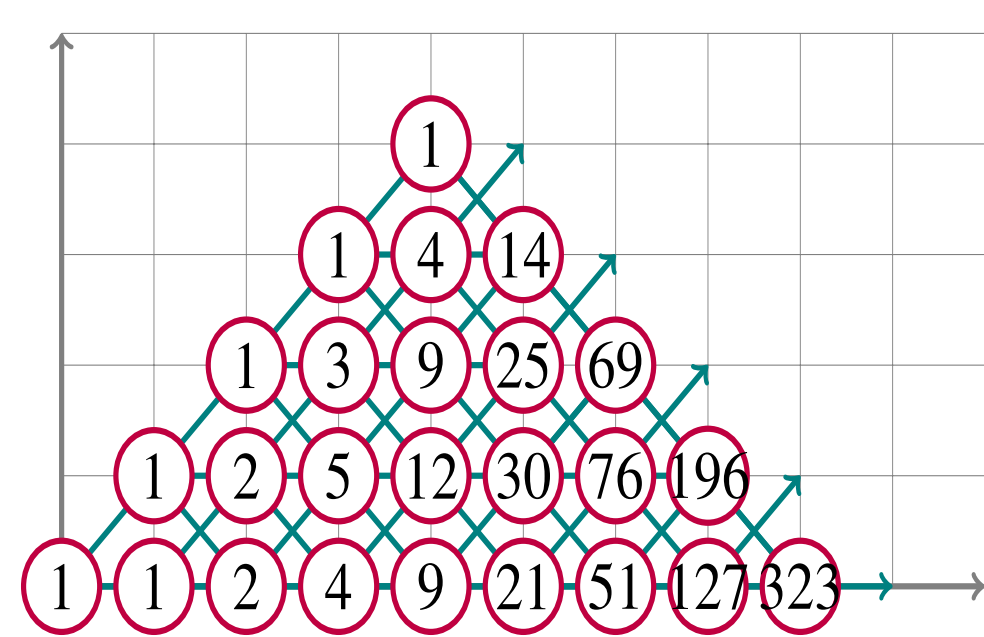$$M_n = 1, 1, 2, 4, 9, 21, 51, 127, \ldots.$$

Figure 2: Path - $M(x)$

A *graph* is comprised of two finite sets called the *nodes* and the *edges*. An *edge* is a line segment that connects any two nodes. Denote a graph as G, its set of nodes V, and its set of edges E. We want to find a way to represent a Motzkin path as a graph. In this research, we tried two different approaches.

Abstract

A *proliferated* cell is a cell that descended from a stem cell, does not have a specific task and can be differentiated to create a specialized cell type. A *differentiated* cell is a descendent from a stem cell that has a special task. When a proliferated cell divides into two cells, the resulting cells are either both proliferated, both differentiated or one of each. Many biologists aim to determine the rate at which a proliferated cell will divide into each scenario. It has been shown that this cell division can be modeled as a random walk that starts at the origin, never goes below the x-axis and ends at the point $(n, 0)$. More specifically, we can model this division using Motzkin paths. This research is focused on finding new ways to visualize the Motzkin paths to determine anymore information about the relationship between the various types of proliferated cell division. In this paper, we will take you through the various representations we created for a Motzkin path and describe any characterizations about the cell divisions these visualizations were able to determine.

Rainbow Bushes

Let M be a Motzkin path. Pick integer n such that M is comprised of n steps. Let V be a collection of nodes with $|V| = n+1$. Write $V = \{v_1, v_2, \ldots, v_{n+1}\}$. Let there be an edge between two nodes v_i and v_j if and only if $i = 1$ and $j = 2, 3, \ldots, n+1$. Let $E = \{E_{1,j} : j = 1, 2, \ldots, n+1\}$ be a collection of edges. Then $|E| = n$. Let G be the graph consisting of V and E. Define function $f : E \to \{1, 2, 3\}$ such that

$$f(E_{1,j}) = \begin{cases} 1, & \text{if the } j-1 \text{ step is } (1,1) \\ 2, & \text{if the } j-1 \text{ step is } (1,0) \\ 3, & \text{if the } j-1 \text{ step is } (1,-1) \end{cases}.$$

This process creates a graph in which there is an edge between every node and the first node, and each edge is one of three colors. We call graphs of this type a *rainbow bush*.
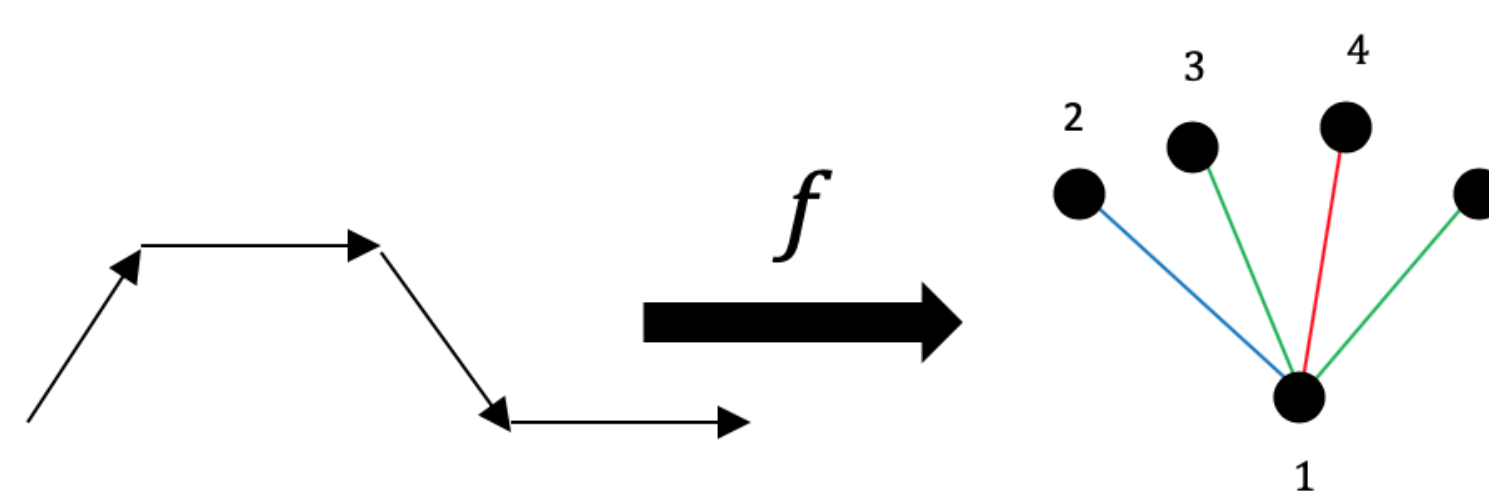

Figure 3: Visual of Motzkin Path with its Rainbow Bush

As a reminder, our goal of this process is to determine any information about the values a, b, and c. Let $\{m\}_f$ be the collection of edges in E such that $f(E_{1,j}) = m$. Note, the image classes under the mapping f partitions the steps used to make M where $\{1\}_f$ contains all of the up steps of M, $\{2\}_f$ contains all the level steps of M and $\{3\}_f$ contains all the down steps of M. Let $u = |\{1\}_f|$, $\ell = |\{2\}_f|$, and $d = |\{3\}_f|$. Then the following is true:

- For each path M, $u + d + \ell = n$.
- Since M starts and ends with $y = 0$ then $u = d$.
- The likelihoods $a + b + c = 1$ which implies $b = 1 - 2a = 1 - 2c$.

This process was nice because it allowed us the ability to isolate the different types of sets, compare the sets of steps, and determine what effect the collection of steps had on the big picture. However, it is desirable to represent the Motzkin paths as complete graphs. (A complete graph is a graph where every two distinct nodes have an edge between them.)

Future Work

We want to further explore the count for number of paths to any one position (n, k). To aid in this cause, we want to look at the Riordan array associated with the Motzkin path and the A-sequence associated with it. The Riordan array will focus more on each step rather than the resulting position.

2-colored Complete Graphs

Let M be a Motzkin path that stops at the point $(n, 0)$. Note that the ith step of M ends at coordinate point (x_i, y_i) with $(x_0, y_0) = (0, 0)$ and $(x_n, y_n) = (n, 0)$. We want to visualize M as a complete graph. Let V be a collection of nodes such that $|V| = n+1$ and write $V = \{1, 2, 3, \ldots, n+1\}$. Let K_n be the complete graph on $|V|$. Let E be the set of edges in K_n and write $E = \{E_{i,j} : i, j = 1, 2, 3, \ldots, n+1\}$. For each $j = 1, 2, \ldots, n+1$, let v_j cooresponds to the position after the previous step, (x_{j-1}, y_{j-1}). Define $f : E \to \{1, 2\}$ such that

$$f(E_{i,j}) = \begin{cases} 1 & \text{if } y_i = y_j \\ 2 & \text{if } y_i \neq y_j \end{cases}.$$
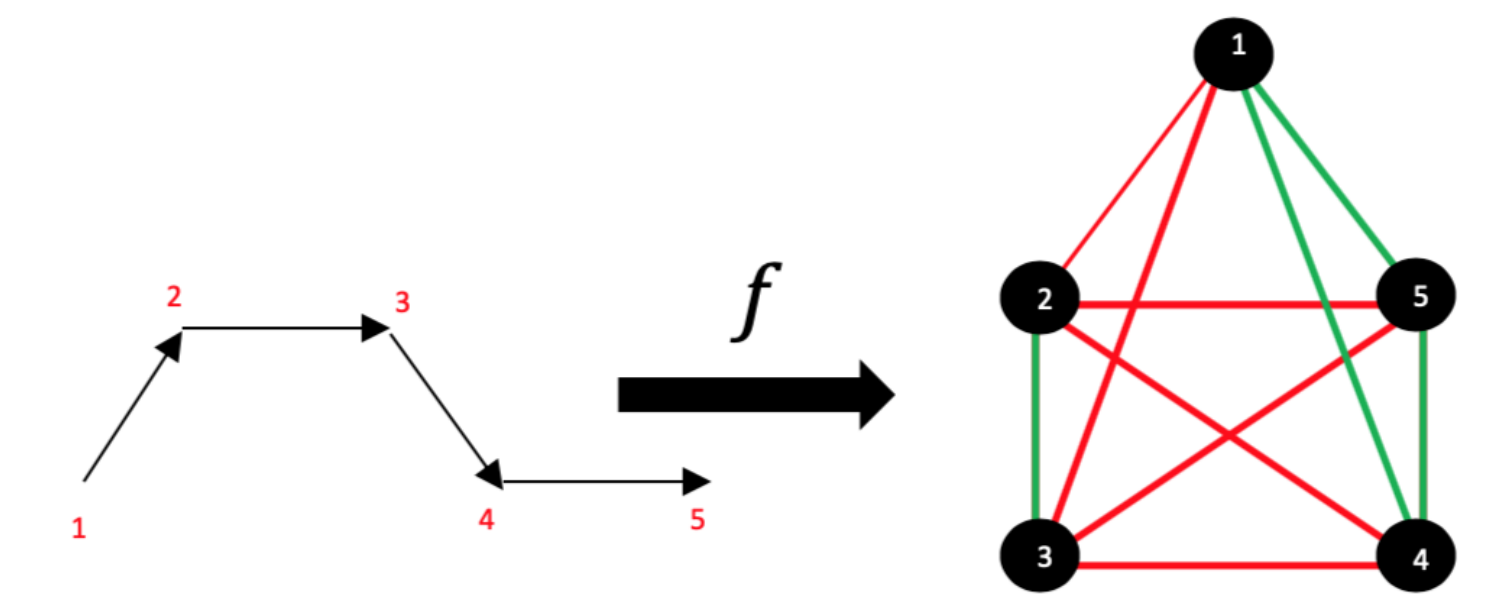

Figure 4: Visual of Motzkin Path with its 2-colored Complete Graph

Since our function maps our edges into one of two image classes, we say that the complete graph K is 2-colored. Let K_M be the 2-colored graph for Motzkin path M and $\mathcal{K}_{Mn}$ be the collection of all 2-colored complete graphs that represent a Motzkin path with n steps. Notice, under this coloring we are grouping together clusters of cells that have the same number of proliferated cells. We were able to determine that we can also use f to define an equivalence relation on the nodes of the graph. Unfortunately, these representation gave similar findings as before because we are requiring that each path must end at $(n, 0)$.

Partial Motzkin Paths

Let (n, k) be any position in the first quadrant of the xy-coordinate plan. How many Motkzin paths, $\mathcal{M}$, end at this point? Towards the end of our summer, we realized it was better to turn our attention to any position. Let u, d, ℓ be the number of up, down, and level steps of $\mathcal{M}$ respectively. Then u no longer has to equal d. Some preliminary results:

- If n is odd then $\ell > 0$.
- The values ℓ and $n - k$ have the same parity.

References

[1] Roshan, A., Jones, P. H., amp; Greenman, C. D. (2014). Exact, time-independent estimation of clone size distributions in normal and mutated cells. Journal of The Royal Society Interface