Classifying toric 3-fold codes of dimensions 4 and 5

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Definition

Let P be an integral convex polytope in $\mathbb{R}^m$. The *toric code* C_P is a vector space $Im(\epsilon)$, where

$$\epsilon : \mathcal{L}(P) \to (\mathbb{F}_q)^m,$$

and $\mathcal{L}(P)$ is the set of polynomials

$$\mathcal{L}(P) = \text{Span}\{\mathsf{x}^p : p \in P \cap \mathbb{Z}^m\}.$$

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Remark

$dim(C_P) = |P \cap \mathbb{Z}^m|$

Example

Let

$$P = Conv\{(0,0,0),(2,0,0),(0,2,0),(0,0,2),\\ (2,2,0),(2,0,2),(0,2,2),(2,2,2)\}$$

Then $dim(C_P) = 27$. Each lattice point $p = (a,b,c) \in P \cap \mathbb{Z}^3$ can be identified with a monomial $\mathbf{x}^p = x^a y^b z^c$.

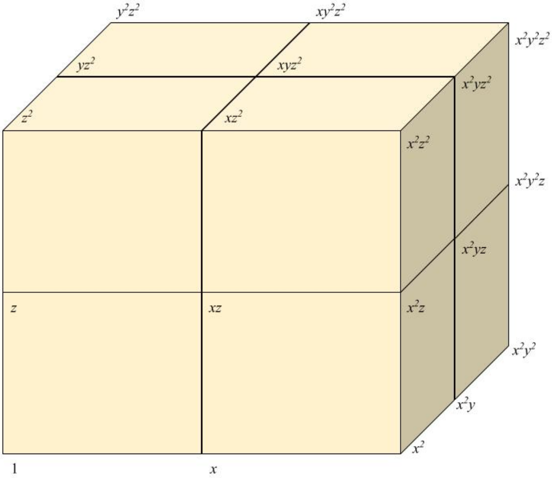

Figure: Monomials on the cube in $\mathbb{Z}^3$

- Hansen introduced toric codes (1998)
- Little and Schwarz classified toric surface codes up to dimension 5 (2007)
- UM-Dearborn REU classified toric surface codes of dimension 7 (2019)
- We will classify codes of dimension 4 beyond the surface.

Definition (Lattice Equivalence)

Two integral convex n-polytopes P_1 and P_2 are *lattice equivalent* if there exists an affine unimodular transformation $t : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $t(P_1) = P_2$.

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Theorem (White)

Every 3-polytope with 4 integral lattice points of volume t is lattice equivalent to an empty tetrahedron

$$T(s,t) = Conv\{(0,0,0),(1,0,0),(0,0,1),(s,t,1)\},$$

for some $s \in \mathbb{Z}$ where $\gcd(s,t) = 1$.

Moreover, $T(s,t)$ is lattice equivalent to $T(s',t)$ if and only if $s' = \pm s^{(\pm 1)}$ (mod t).

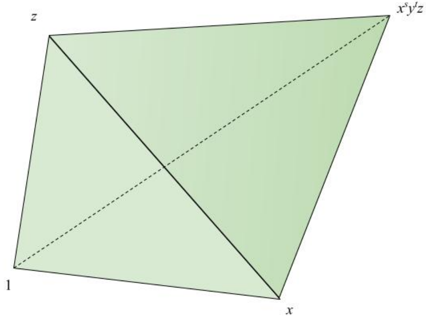

Figure: Monomials on the empty tetrahedron

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The *minimum distance* of a code C is

$$d(C) = (q-1)^m - \max_{0 \neq f \in \mathcal{L}(P)} Z(f)$$

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Theorem (Minimum Distance on Empty Tetrahedra)

Let C be a toric code over $\mathbb{F}_q$ on an empty tetrahedron $T(s,t)$. Then

- $d(C) = (q-1)^3 - (q-1)(q-3) - 2(q-1)$ *if* $\gcd(t, q-1) = 1$, *and*
- $d(C) = (q-1)^3 - (q-1)(q-3) - q\gcd(t, q-1)$ *otherwise.*

Definition (Generator Matrix)

If C_P is of length n and dimension k, then a generator matrix G is a $k \times n$ matrix with rows given by $p \in P$ and columns given by $\mathbf{x} \in (\mathbb{F}_q^*)^m$.

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Definition (Monomial Equivalence)

Let C_1 and C_2 be of length n over $\mathbb{F}_q$. Let G_1 be a generator matrix for C_1. Then C_1 and C_2 are *monomially equivalent* if there is an invertible $n \times n$ diagonal matrix Δ and an $n \times n$ permutation matrix Π such that

$$G_2 = G_1 \Delta \Pi$$

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Remark (Little and Schwarz)

Lattice equivalence implies monomial equivalence.

Theorem (Monomial Equivalence Theorem 1)

Let C_1 and C_2 be toric codes with four lattice points on empty tetrahedra $T(s, t_1)$ and $T(s, t_2)$, respectively, over the field $\mathbb{F}_q$. Then C_1 and C_2 are monomially equivalent iff $\gcd(t_1, q-1) = \gcd(t_2, q-1)$.

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Proof sketch. ($\Rightarrow$) We apply our results for minimum distance. Two codes with different minimum distances cannot be monomially equivalent.

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Proof sketch. ($\Rightarrow$) We apply our results for minimum distance. Two codes with different minimum distances cannot be monomially equivalent.

($\Leftarrow$) We show a permutation between generator matrices exists by demonstrating a bijective correspondence between columns of the two matrices.

Theorem (Monomial Equivalence Theorem 2)

Let C_1 and C_2 be toric codes over $\mathbb{F}_q$ on empty tetrahedra $T(s_1, t)$ and $T(s_2, t)$, respectively. Then C_1 and C_2 are monomially equivalent iff either of the following conditions hold true:

1. $s_1 \equiv s_2 \bmod \gcd(t, q-1)$, *or*
2. $s_1 \equiv \pm s_2^{\pm 1} \bmod t$.

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Case 2: We show explicitly that the two polytopes are lattice equivalent.

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Proof sketch. ($\Rightarrow$) Case 1: Using a primitive element, we show that the generator matrices are equivalent.

Case 2: We show explicitly that the two polytopes are lattice equivalent.

($\Leftarrow$) We argue by contradiction that there exists no permutation between generator matrices.

1. ~~Introduction~~
2. ~~A minimum distance formula~~
3. ~~Two classification theorems~~
4. Toric codes of dimension 5

Sig.	Representative
(2,2)	(0,0,0),(1,0,0),(0,1,0),(1,1,0),(0,0,1)
(2,1)	(0,0,0),(1,0,0),(-1,0,0),(0,0,1),(s,t,1) $0 \leq s \leq \frac{t}{2}, \gcd(s,t) = 1$
(3,2)	(0,0,0),(1,0,0),(0,1,0),(0,0,1),(s,t,1) $0 < \mathrm{s} \leq t, \gcd(s,t) = 1$
(3,1)	(0,0,0),(1,0,0),(0,1,0),(0,0,1),(-1,-1,0)

Table: Five point 3-polytope classes of width 1.

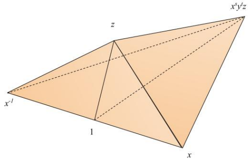

Figure: Signature (2,1).

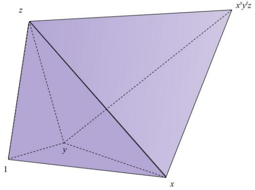

Figure: Signature (3,2).

Proposition (Minimum distance for 5 dimensions)

1. *If* $sig(P) = (2,1)$, *then* $d = (q-1)^3 - 2(q-1)^2$;
2. *If* $sig(P) = (2,2)$, *then* $d = (q-1)^3 - (2q^2 - 5q + 3)$;
3. *If* $sig(P) = (3,1)$ *and* P *has width 1, then*
$$d \geq (q-1)^3 - (q-1)(1 + q + 2\sqrt{q}); \text{ and}$$
4. *If* $sig(P) = (3,2)$, *then*
$$d \geq (q-1)^3 - (q-1)^2 - (s+t)q$$

Remark

For codes over polytopes of width 2, we can take the minimum distance of the subpolytope empty tetrahedron as a lower bound on d, *and hope to improve on these.*

- We have proof of a full classification of toric 3-fold codes of 4 dimensions.
- We have some information about the minimum distances of toric 3-fold codes of 5 dimensions.
- Further work is needed on the width 2 polytopes to complete the 5 dimension classification.

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