Generalizing Alder's conjecture via a conjecture of Kang and Park

Adriana Duncan[1], Simran Khunger[2]

Collaborator: Ryan Tamura[3], Advisor: Dr. Holly Swisher[4]

[1]Tulane University

[2]Carnegie Mellon University

[3]University of California, Berkeley

[4]Oregon State University

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   - Partition Identities: Past and Present
   - The Kang-Park Conjecture

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What is a Partition of n?

Definition

A **partition** of a *positive* integer n is a *non-increasing* sequence of *positive* integers $(a_1, a_2, ..., a_r)$, such that $\sum_{i=1}^{r} a_i = n$. Let $p(n \,|\, \text{condition})$ be the number of partitions of n satisfying a certain condition.

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Example

Let $n = 4$, the partitions of 4 are
$1+1+1+1$
$2+1+1$
$2+2$
$3+1$
4
Thus, $p(4 \mid \text{unrestricted}) = 5$.

Notation and Preliminaries

Following Kang and Park, we will denote

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Definition

Let $\Delta_d^{(a,b)}(n) := q_d^{(a)}(n) - Q_d^{(b)}(n)$ and $\Delta_d^{(a)}(n) := \Delta_d^{(a,a)}(n)$

Notation Example: Partition Identities

Using the notation we have above

Example

$$\text{Euler's identity is } \Delta_1^{(1)}(n) = q_1^{(1)}(n) - Q_1^{(1)}(n) = 0 \text{ for all } n > 0$$

$$\text{R-R (1st identity) is } \Delta_2^{(1)}(n) = q_2^{(1)}(n) - Q_2^{(1)}(n) = 0 \text{ for all } n > 0$$

$$\text{R-R (2nd identity) is } \Delta_2^{(2)}(n) = q_2^{(2)}(n) - Q_2^{(2)}(n) = 0 \text{ for all } n > 0$$

$$\text{Schur's identity } \Rightarrow \Delta_3^{(1)}(n) = q_3^{(1)}(n) - Q_3^{(1)}(n) \geq 0 \text{ for all } n > 0$$

A Recap of Alder's Conjecture

In 1956, Alder conjectured that

Conjecture (Alder, 1956)

$$q_d^{(1)}(n) - Q_d^{(1)}(n) = \Delta_d^{(1)}(n) \geq 0 \textit{ for all } n, d > 0.$$

The Euler, first Rogers-Ramanujan, and Schur identities are special cases of Alder's conjecture.

Proving Alder's Conjecture

Theorem (Andrews, 1971)

$\Delta_d^{(1)}(n) \geq 0$ *for all* $n > 0$ *and* $d = 2^s - 1$ *for* $s \geq 4$.

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Theorem (Yee, 2008)

$\Delta_d^{(1)}(n) \geq 0$ *for all* $n > 0$ *and* $d \geq 32$ *or* $d = 7$.

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Theorem (Yee, 2008)

$$\Delta_d^{(1)}(n) \geq 0 \textit{ for all } n > 0 \textit{ and } d \geq 32 \textit{ or } d = 7.$$

Theorem (Alfes, Jameson, Lemke Oliver, 2011)

$$\Delta_d^{(1)}(n) \geq 0 \textit{ for all } n > 0 \textit{ and } 4 \leq d \leq 30, d \neq 7, 15.$$

The Kang-Park Conjecture

In 2020, Kang and Park conjectured a generalization of the second Rogers-Ramanujan identity.

Definition

$Q_d^{(b,-)}(n)$ counts the number of partitions of n into parts $\equiv \pm b \mod (d+3)$ excluding the part $d+3-b$.

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Conjecture (Kang and Park's Conjecture)

$$q_d^{(2)}(n) - Q_d^{(2,-)}(n) =: \Delta_d^{(2,-)}(n) \geq 0 \text{ for all } n, d > 0.$$

Kang and Park's Results

Theorem (Kang and Park, 2020)

Let $d = 2^s - 2$ for $s = 2$ or $s \geq 5$. Then for any positive even integer n, we have

$$\Delta_d^{(2,-)}(n) \geq 0$$

Project Goals

Conjecture (Kang and Park's Conjecture, 2020)

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$$q_d^{(2)}(n) - Q_d^{(2,-)}(n) = \Delta_d^{(2,-)}(n) \geq 0 \text{ for all } n, d > 0.$$

Definition

Let $1 \leq b \leq d+2$. *Then define*

$Q_d^{(b,-,-)}(n)$ *as* $p(n \mid \text{parts} \equiv \pm b \pmod{d+3}, \text{parts} \neq b \text{ or } d+3-b)$.

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Conjecture (Generalized Kang and Park Conjecture)

If $1 \leq a \leq d+2$ *and* d *and* n *be positive integers, then*

$$\Delta_d^{(a,-,-)}(n) = q_d^{(a)}(n) - Q_d^{(a,-,-)}(n) \geq 0$$

Preliminary Theorems

Lemma (Generalized Andrews' Lemma)

Let $S = \{x_i\}_{i=1}^{\infty}$ and $T = \{y_i\}_{i=1}^{\infty}$ be increasing sequences of positive integers with each y_i divisible by m, such that $y_1 = m$ and $x_i \geq y_i$ for all i. Then for all $n > 0$,

$$\rho(T; mn) \geq \rho(S; mn) \text{ for all } mn.$$

where $\rho(S; n)$ (resp. $\rho(T; n)$) denotes the number of partitions of n into parts taken from S (resp. T).

Modifications of Alder's Conjecture

Lemma (1)

For any $k \geq 32$,

$$q_k^{(1)}(m) \geq Q_{k-2}^{(1,-)}(m)$$

for all positive m.

Proof. From Yee, we have the inequality

$$q_k^{(1)}(m) \geq g_k^{(1)}(m).$$

Using the generalized Andrews' lemma we can show

$$g_k^{(1)}(m) \geq Q_{k-2}^{(1,-)}(m).$$

□

Modifications of Alder's Conjecture

Lemma (2)

For any $k \geq 32$,

$$q_k^{(1)}(m) \geq Q_{k-3}^{(1,-,-)}(m)$$

for all positive m.

Proof. From Yee, we have the inequality

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$$g_k^{(1)}(m) \geq Q_{k-3}^{(1,-,-)}(m).$$

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Proof of Kang and Park's Conjecture

Theorem

For $d \geq 62$ and $n > 0$,

$$\Delta_d^{(2,-)}(n) = q_d^{(2)}(n) - Q_d^{(2,-)}(n) \geq 0.$$

We prove this in four cases based on the parity of d and n.

- **Case 1:** d odd, n odd.
- **Case 2:** d odd, n even.
- **Case 3:** d even, n even.
- **Case 4:** d even, n odd.

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- **Case 4:** d even, n odd.

Notice Case 1 is trivial since

$$Q_d^{(2,-)}(n) = p(n|\text{parts } \equiv \pm 2 \pmod{d+3},\ \text{part} \neq d+1) = 0$$

d odd, n even

We prove the following inequality chain:

injection bijection

$$q_d^{(2)}(n) \geq q_{\frac{d+1}{2}}^{(1)}\left(\frac{n}{2}\right) \geq Q_{\frac{d-3}{2}}^{(1,-)}\left(\frac{n}{2}\right) = Q_d^{(2,-)}(n)$$

Lemma 1

Lemma (1)

For any $k \geq 32$,

$$q_k^{(1)}(m) \geq Q_{k-2}^{(1,-)}(m)$$

for all positive m.

Completing Kang and Park's conjecture

We prove the following inequality chain for d even, n even:

$$q_d^{(2)}(n) \geq q_{\frac{d}{2}}^{(1)}\left(\frac{n}{2}\right) \geq Q_{\frac{d}{2}-2}^{(1,-)}\left(\frac{n}{2}\right) = Q_{d-1}^{(2,-)}(n) \geq Q_d^{(2,-)}(n)$$

injection; Lemma 1; bijection; generalized Andrews' lemma

We prove the following inequality chain for d even, n odd:

$$q_d^{(2)}(n) \geq q_{\frac{d}{2}}^{(1)}\left(\frac{n+1}{2}\right) \geq Q_{\frac{d}{2}-2}^{(1,-)}\left(\frac{n+1}{2}\right) = Q_{d-1}^{(2,-)}(n+1) \geq Q_d^{(2)}(n)$$

injection; Lemma 1; bijection; injection

Recall the Generalized Kang and Park Conjecture

Definition

Let $1 \leq b \leq d+2$. *Then define*

$Q_d^{(b,-,-)}(n)$ *as* $p(n|$ *parts* $\equiv \pm b \pmod{d+3}$, *parts* $\neq b$ *or* $d+3-b)$.

Conjecture (Generalized Kang and Park Conjecture)

If $1 \leq a \leq d+2$ *and* d *and* n *be positive integers, then*

$$\Delta_d^{(a,-,-)}(n) = q_d^{(a)}(n) - Q_d^{(a,-,-)}(n) \geq 0.$$

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If $1 \leq a \leq d+2$ *and* d *and* n *be positive integers, then*

$$\Delta_d^{(a,-,-)}(n) = q_d^{(a)}(n) - Q_d^{(a,-,-)}(n) \geq 0.$$

Notice that if $d+3$ is divisible by a and n is not divisible by a, $Q_d^{(a,-,-)}(n) = 0$ and $\Delta_d^{(a,-,-)}(n) \geq 0$ trivially.

Partially Proving the Generalized Kang and Park Conjecture

Theorem (Generalized Kang and Park Conjecture, Thm. 1)

If $d+3$ is divisible by a, then for all $n > 0$

$$\Delta_d^{(a,-,-)}(n) = q_d^{(a)}(n) - Q_d^{(a,-,-)}(n) \geq 0.$$

Thm. 1: Reduction Method

We prove the following chain of inequalities for $d+3$ and n divisible by a:

injection

bijection

$$q_d^{(a)}(n) \geq q_{\frac{d+3}{a}}^{(1)}\left(\frac{n}{a}\right) \geq Q_{\frac{d+3}{a}-3}^{(1,-,-)}\left(\frac{n}{a}\right) = Q_d^{(a,-,-)}(n)$$

Lemma 2

Lemma (2)

For any $k \geq 32$,

$$q_k^{(1)}(m) \geq Q_{k-3}^{(1,-,-)}(m)$$

for all positive m.

Partially Proving the Generalized Kang and Park Conjecture

Define r as the largest integer such that $2^r - 2^{a-1} \leq d$.

Theorem (Generalized Kang and Park Conjecture, Thm. 2)

If $d \geq 31 \cdot 2^{a-1}$*, and* $n \geq 4d + 2^r$ *where* d *and* n *are divisible by* 2^{a-1}*, then*

$$\Delta_d^{(a,-,-)}(n) = q_d^{(a)}(n) - Q_d^{(a,-,-)}(n) \geq 0.$$

Thm. 2: Andrews' and Yee's Method

Following Andrews, for any $d = 2^s - 2^t$, with $s \geq t+4$, and n divisible by 2^t we have

$$q_d^{(a)}(n) \geq \mathscr{L}_d^{(a)}(n) = \rho(T;n) \geq Q_d^{(a,-,-)}(n)$$

Following Yee, for $d \geq 2^{a-1} \cdot 31$ such that d is divisible by 2^{a-1} and $d \neq 2^r - 2^{a-1}$ and for all $n \geq 4d + 2^r$ where n is divisible by 2^{a-1} we have

$$q_d^{(a)}(n) \geq \mathscr{L}_d^{(a)}(n-2^r) + \mathscr{L}_d^{(a)}(n) \geq K_d^{(a)}(n) \geq G_d^{(a)}(n) \geq Q_d^{(a,-,-)}(n)$$

Remaining cases: generalizing Alfes, Jameson, and Lemke Oliver

Theorem ($\Delta_d^{(a)}(n)$ asymptotic)

We have for $d \geq 4$, $n > 0$, $0 < a < \frac{d+3}{2}$, and a co-prime to $d+3$,

$$\Delta_d^{(a)}(n) = \frac{A^{1/4}}{2\sqrt{\pi\alpha^{d+1-2a}(d\alpha^{d-1}+1)}} n^{-3/4} e^{2\sqrt{An}} + \mathscr{E}_d(n)$$

where $\mathscr{E}_d(n) = r_d(n) - Q_d^{(a)}(n)$.

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where $\mathscr{E}_d(n) = r_d(n) - Q_d^{(a)}(n)$.

Corollary

Let $n > 0$, $a > 0$, and $d \geq 4$, such that $a < \frac{d+3}{2}$ and a is relatively prime to $d+3$, then

$$\lim_{n\to\infty} \Delta_d^{(a,-,-)}(n) = +\infty.$$

References

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