Not Your Normal Fibonacci Sequence

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Background

- The Fibonacci sequence is perhaps the most well-known sequence in the field of mathematics.

- The Fibonacci numbers appear in the number of petals on a flower, division of tree branches, reproductive dynamics, and DNA molecules [2].

  0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots
We will generalize the Fibonacci sequence.

We will denote the first two values in the series are $a$ and $b$ where $a, b \in \mathbb{N}$ and $0 \leq a < b$

$a, b, a + b, a + 2b, 2a + 3b, 3a + 5b, 5a + 8b, 8a + 13b, \cdots$
Objective:

Given a number $n$ we seek to find the smallest positive value $b$ such that $n$ appears in the Fibonacci series starting with $a$ and $b$. 
Example

- Consider the case where \( n = 27 \).

- The smallest positive value \( b \) such that \( n \) appears in the sequence occurs when \( a = 3 \) and \( b = 7 \).

- The sequence is as follows:

\[ 3, 7, 10, 17, 27, \ldots \]
This problem can be solved with an exhaustive search for various combinations of $a$ and $b$.

We investigate more efficient methods to determine $a$ and $b$ than by brute force.

We are only interested in minimizing the value of $b$ but not the value of $a$. This is because the smallest $a$ value will always be 0.
Objective:

Given a number $n$ we seek to find the smallest positive value $b$ such that $n$ appears in the Fibonacci series starting with $a$ and $b$. 
Question:

When searching for the smallest $b$ value does there exist a stricter relationship between the $a$ and $b$ values, apart from $a < b$?
By inspection, we notice that not all of the possible values of $a$ less than $b$ appear for various $n$ values.

For example, when $b = 10$, the possible corresponding $a$ values are $a = 0$, $a = 1$, $a = 2$, $a = 3$, and $a = 4$.

Further inspection of possible values of $a$ for various $b$ values leads us to determine that $a < \frac{b}{2}$. 
Consider the case where \( a = 4 \) and \( b = 5 \). Then our pattern is as follows:

\[ 4, 5, 9, 14, 23, 37, \cdots \]

However, this can be written as

\[ 1, 4, 5, 9, 14, 23, 37, \cdots \]

where now we have \( a = 1 \) and \( b = 4 \).
Motivation

Conjectures

Conclusion

Method

Theorem 1:

If $a, b, n \in \mathbb{N}$ and $n$ appears in the Fibonacci series starting with $a$ and $b$ where $b$ is the smallest positive value such that $n$ appears in the series, then $a < \frac{b}{2}$.
Objective:

Is there a faster way to determine the minimal $b$ value other than by a brute-force search?
If $n = 100$, then we can write $n$ using two numbers as

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
<td>99</td>
</tr>
<tr>
<td>2</td>
<td>98</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

and so on.

The left value represents our $a$ value, and the right value represents our $b$ value.

We generate valid sequences until $a \geq \frac{b}{2}$.
We have

\[
\begin{array}{cc}
0 & 100 \\
(+) & (-) \\
1 & 99 \\
\vdots & \vdots \\
\end{array}
\]

and so on... So we have the following inequality,

\[0 + (1)s \geq \frac{100 - (1)s}{2}\]

\[s \geq 33.3\]

\[s \geq 34\]

where \(s\) is the number of steps. This works until we reach the 34th step or when \(a = 34\) and \(b = 66\).
We examine the first case where our condition is violated. Now we have

\[
\begin{array}{ccccc}
2 & 32 & 34 & 66 \\
(+3) & (-2) & (+1) & (-1) \\
5 & 30 & 35 & 65 \\
: & : & : & :
\end{array}
\]

and so on... Similarly to before we have the following inequality,

\[
2 + (3)s \geq \frac{32 - (2)s}{2}
\]

\[
s \geq 3.5
\]

\[
s \geq 4
\]

Now we have \( a = 14 \) and \( b = 24 \).
We repeat this process until we have the last case where our condition is not violated.

<table>
<thead>
<tr>
<th></th>
<th>4</th>
<th>10</th>
<th>14</th>
<th>24</th>
<th>38</th>
<th>62</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(+8)</td>
<td>(-5)</td>
<td>(+3)</td>
<td>(-2)</td>
<td>(+1)</td>
<td>(-1)</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>17</td>
<td>22</td>
<td>39</td>
<td>61</td>
<td></td>
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<td></td>
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</tbody>
</table>

and so on...

Therefore, for \( n = 100 \), our values are \( a = 4 \) and \( b = 10 \).
Question:

Is there a faster way to determine $a$ and $b$ for certain $n$ values?
We will first explore various $n$ values that are the product of two Fibonacci numbers and examine their corresponding values of $a$ and $b$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5 = (1 \times 5)$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$9 = (3 \times 3)$</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>$15 = (3 \times 5)$</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>$40 = (5 \times 8)$</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>$68 = (2 \times 34)$</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

From the table above, we are able to see that for each value of $n$, $a = 0$ and $b$ are equal to the smaller Fibonacci number involved the product.
Conjecture 1:

If \( n \) is the product of two Fibonacci numbers, \( f_i \) and \( f_j \), such that \( i \leq j \), then \( a = 0 \) and \( b = f_i \).
Now we will explore the problem when every factor pair of \( n \) is of the form, \( n = f \times k \), where \( f \) is a Fibonacci number and \( k \) is not.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( a )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12 = (3 \times 4)</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>30 = (5 \times 6)</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>35 = (5 \times 7)</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>52 = (13 \times 4)</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>117 = (13 \times 9)</td>
<td>0</td>
<td>9</td>
</tr>
</tbody>
</table>

From the table above, we are able to see that for each value of \( n \), \( a = 0 \) and \( b = k \) in the factor pair with the largest \( f \).
Conjecture 2:

If every factor pair of $n$ is of the form, $n = f \times k$, where $f$ is a Fibonacci number and $k$ is not, we choose the factor pair with the largest $f$. Then $a = 0$ and $b = k$. 
**Question:**

Does every positive value have at most one factor pair with two Fibonacci numbers?
Products of Fibonacci Numbers

- The answer is yes!

- Carmichael’s Theorem illustrates that after the number 144 in the Fibonacci sequence there exists at least one prime divisor that does not divide any of the previous Fibonacci numbers [1].

- We can look at all of the Fibonacci numbers before and including 144 which are

\[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144\]

and see that none of the values are a result of the product of two Fibonacci numbers apart from 1 and itself.
Products of Fibonacci Numbers

- Thus there do not exist two Fibonacci numbers whose product is another Fibonacci number.

- We can extend this idea to show that every non-Fibonacci number is the product of at most two Fibonacci numbers.
**Conjecture 3:**

*Every positive value has at most one factor pair whose values are both Fibonacci numbers.*
Conclusion

- We proved that $a < \frac{b}{2}$.
- We developed a method determining for determining $a$ and $b$.
- We proposed 3 conjectures.
Conjectures

- **Conjecture 1:** If \( n \) is the product of two Fibonacci numbers, \( f_i \) and \( f_j \), such that \( i \leq j \), then \( a = 0 \) and \( b = f_i \).

- **Conjecture 2:** If every factor pair of \( n \) is of the form, \( n = f \times k \), where \( f \) is a Fibonacci number and \( k \) is not, we choose the factor pair with the largest \( f \). Then \( a = 0 \) and \( b = k \).

- **Conjecture 3:** Every positive value has at most one factor pair whose values are both Fibonacci numbers.
Future Work

- It is paramount to formalize a proof for the conjectures and our methodical approach.
- We are interested in investigating if there is a faster way to determine $a$ and $b$ for $n$ values not covered by the proposed conjectures.
- We want to look into determining the first $n$ value where a specific $b$ value occurs.

Thank You!