Not Your Normal Fibonacci Sequence

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Background

- The Fibonacci sequence is perhaps the most well-known sequence in the field of mathematics.
- The Fibonacci numbers appear in the number of petals on a flower, division of tree branches, reproductive dynamics, and DNA molecules [2].

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \cdots$$

Background

- We will generalize the Fibonacci sequence.
- We will denote the first two values in the series are a and b where $a, b \in \mathbb{N}$ and $0 \leq a < b$

$a, b, a + b, a + 2b, 2a + 3b, 3a + 5b, 5a + 8b, 8a + 13b, \cdots$

Objective:

Given a number n we seek to find the smallest positive value b such that n appears in the Fibonacci series starting with a and b.

Example

- Consider the case where $n = 27$.
- The smallest positive value b such that n appears in the sequence occurs when $a = 3$ and $b = 7$.
- The sequence is as follows:

$$3, 7, 10, 17, 27, \cdots$$

Background

- This problem can be solved with an exhaustive search for various combinations of a and b.
- We investigate more efficient methods to determine a and b than by brute force.
- We are only interested in minimizing the value of b but not the value of a. This is because the smallest a value will always be 0.

Objective:

Given a number n we seek to find the smallest positive value b such that n appears in the Fibonacci series starting with a and b.

Question:

When searching for the smallest b value does there exist a stricter relationship between the a and b values, apart from $a < b$?

Relationship between a and b

- By inspection, we notice that not all of the possible values of a less than b appear for various n values.
- For example, when $b = 10$, the possible corresponding a values are $a = 0$, $a = 1$, $a = 2$, $a = 3$, and $a = 4$.
- Further inspection of possible values of a for various b values leads us to determine that $a < \frac{b}{2}$.

Example

Consider the case where $a = 4$ and $b = 5$. Then our pattern is as follows:

$$4, 5, 9, 14, 23, 37, \cdots$$

However, this can be written as

$$1, 4, 5, 9, 14, 23, 37, \cdots$$

where now we have $a = 1$ and $b = 4$.

Theorem 1:

If $a, b, n \in \mathbb{N}$ *and* n *appears in the Fibonacci series starting with* a *and* b *where* b *is the smallest positive value such that* n *appears in the series, then* $a < \frac{b}{2}$.

Objective:

Is there a faster way to determine the minimal b value other than by a brute-force search?

If $n = 100$, then we can write n using two numbers as

$$\begin{array}{cc} 0 & 100 \\ 1 & 99 \\ 2 & 98 \\ \vdots & \vdots \end{array}$$

and so on.

The left value represents our a value, and the right value represents our b value.

We generate valid sequences until $a \geq \frac{b}{2}$.

We have

$$\begin{array}{cc} 0 & 100 \\ (+1) & (-1) \\ 1 & 99 \\ \vdots & \vdots \end{array}$$

and so on... So we have the following inequality,

$$\begin{aligned} 0 + (1)s \geq & \frac{100 - (1)s}{2} \\ s \geq & 33.\overline{3} \\ s \geq & 34 \end{aligned}$$

where s is the number of steps. This works until we reach the 34th step or when $a = 34$ and $b = 66$.

We examine the first case where our condition is violated. Now we have

2	32	34	66
(+3)	(-2)	(+1)	(-1)
5	30	35	65
⋮	⋮	⋮	⋮

and so on... Similarly to before we have the following inequality,

$$2 + (3)s \geq \frac{32 - (2)s}{2}$$
$$s \geq 3.5$$
$$s \geq 4$$

Now we have $a = 14$ and $b = 24$.

We repeat this process until we have the last case where our condition is not violated.

4	10	14	24	38	62
(+8)	(-5)	(+3)	(-2)	(+1)	(-1)
12	5	17	22	39	61
⋮	⋮	⋮	⋮	⋮	⋮

and so on...

Therefore, for $n = 100$, our values are $a = 4$ and $b = 10$.

Question:

Is there a faster way to determine a and b for certain n values?

n is the Product of Two Fibonacci Numbers

We will first explore various n values that are the product of two Fibonacci numbers and examine their corresponding values of a and b.

n	a	b
$5 = (1 \times 5)$	0	1
$9 = (3 \times 3)$	0	3
$15 = (3 \times 5)$	0	3
$40 = (5 \times 8)$	0	5
$68 = (2 \times 34)$	0	2

From the table above, we are able to see that for each value of n, $a = 0$ and b are equal to the smaller Fibonacci number involved the product.

Conjecture 1:

If n is the product of two Fibonacci numbers, f_i and f_j, such that $i \leq j$, then $a = 0$ and $b = f_i$.

n has a Fibonacci Number in Every Factor Pair

Now we will explore the problem when every factor pair of n is of the form, $n = f \times k$, where f is a Fibonacci number and k is not.

n	a	b
$12 = (3 \times 4)$	0	4
$30 = (5 \times 6)$	0	6
$35 = (5 \times 7)$	0	7
$52 = (13 \times 4)$	0	4
$117 = (13 \times 9)$	0	9

From the table above, we are able to see that for each value of n, $a = 0$ and $b = k$ in the factor pair with the largest f.

Conjecture 2:

If every factor pair of n *is of the form,* $n = f \times k$*, where* f *is a Fibonacci number and* k *is not, we choose the factor pair with the largest* f*. Then* $a = 0$ *and* $b = k$*.*

Question:

Does every positive value have at most one factor pair with two Fibonacci numbers?

- The answer is yes!
- Carmichael's Theorem illustrates that after the number 144 in the Fibonacci sequence there exists at least one prime divisor that does not divide any of the previous Fibonacci numbers [1].
- We can look at all of the Fibonacci numbers before and including 144 which are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$$

and see that none of the values are a result of the product of two Fibonacci numbers apart from 1 and itself.

- Thus there do not exists two Fibonacci numbers whose product is another Fibonacci number.
- We can extend this idea to show that every non-Fibonacci number is the product of at most two Fibonacci numbers.

Conjecture 3:

Every positive value has at most one factor pair whose values are both Fibonacci numbers.

Conclusion

- We proved that $a < \dfrac{b}{2}$.
- We developed a method determining for determining a and b.
- We proposed 3 conjectures.

Conjectures

- **Conjecture 1:** *If n is the product of two Fibonacci numbers, f_i and f_j, such that $i \leq j$, then $a = 0$ and $b = f_i$.*
- **Conjecture 2:** *If every factor pair of n is of the form, $n = f \times k$, where f is a Fibonacci number and k is not, we choose the factor pair with the largest f. Then $a = 0$ and $b = k$.*
- **Conjecture 3:** *Every positive value has at most one factor pair whose values are both Fibonacci numbers.*

Future Work

- It is paramount to formalize a proof for the conjectures and our methodical approach.
- We are interested in investigating if there is a faster ways to determine a and b for n values not covered by the proposed conjectures.
- We want to look into determining the first n value where a specific b value occurs.

References

"Carmichael's Theorem." *Wikipedia*, Wikimedia Foundation, 17 June 2020, en.wikipedia.org/wiki/Carmichael's_theorem.

Dvorsky, George. "15 Uncanny Examples of the Golden Ratio in Nature." *io9*, 20 Feb. 2013, io9.gizmodo.com/15-uncanny-examples-of-the-golden-ratio-in-nature-5985588.

Thank You!