Jacobian Groups of Bipartite Graphs

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A Little Motivation

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Jacobian Groups of Bipartite Graphs
Defining Graphs

Figure: Three Equivalent Graphs

- What matters is what is connected to what
- No coordinate system, unlike in calculus
Some Common Graphs

[Images of three common graphs drawn in blue ink]

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Jacobian Groups of Bipartite Graphs
What is the Jacobian group?

The Jacobian group of every finite graph can be written as the Cartesian product of $\mathbb{Z}_i$’s for some (often multiple) values of $i \in \mathbb{N}$. 
Example: Jacobian Groups of Some Actual Graphs

Revisiting the graphs from before, with their Jacobian groups:

Figure: $\mathbb{Z}_5$  \hspace{2cm} $\mathbb{Z}_5^3$  \hspace{2cm} $\mathbb{Z}_3^2 \times \mathbb{Z}_9$
Example: Jacobian Groups of Some Actual Graphs

Revisiting the graphs from before, with their Jacobian groups:

Figure: $\mathbb{Z}_5$  
$\mathbb{Z}_3^3$  
$\mathbb{Z}_3^2 \times \mathbb{Z}_9$

Pattern: $\mathbb{Z}_n$  
$\mathbb{Z}_n^{n-2}$  
$\mathbb{Z}_n^{2n-4} \times \mathbb{Z}_{n^2}$
Is this useful for anything!?

- The Matrix-Tree Theorem tells us that the size of the Jacobian group of a graph is the number of spanning trees on that graph.
Spanning Trees

Figure: All 3 spanning trees of a graph on 4 vertices
Spanning Trees: Your Turn!

**Figure:** Can you find any of this graph’s spanning trees?
Spanning Trees: Your Turn!

Figure: All 5 spanning trees of $C_5$
Bipartite Graphs

Figure: Bipartite Graph

Figure: $K_{3,3}$

$Jac(K_{n,n}) \cong \mathbb{Z}_n^{2n-4} \times \mathbb{Z}_n^2$
What happens to the Jacobian group if we take a graph and specify a single direction for some edge(s)?
The $B_n$ Graph

**Figure:** The $B_3$ Graph
Theorem

$\text{Jac}(B_n) \cong \mathbb{Z}_n^{2n-4}$ for $n \geq 2$

Figure: $\text{Jac}(B_2) \cong \mathbb{Z}_1$

Figure: $\text{Jac}(B_3) \cong \mathbb{Z}_3^2$

Figure: $\text{Jac}(B_4) \cong \mathbb{Z}_4^4$
Ideas From My Proof

- Counting spanning trees
- Directed spanning trees rooted at a given vertex
- Linear algebra: The Matrix-Tree Theorem
- Group theory: Factor groups; Short exact sequences
A next goal: Generalizing this pattern to take off any number of outgoing edges from one vertex

Figure: \( \text{Jac}(G) \cong \mathbb{Z}_4^3 \times \mathbb{Z}_8 \). What is this pattern?
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