FINDING MINIMUM DOMINATING SETS ON THREE-DIMENSIONAL GRID GRAPHS

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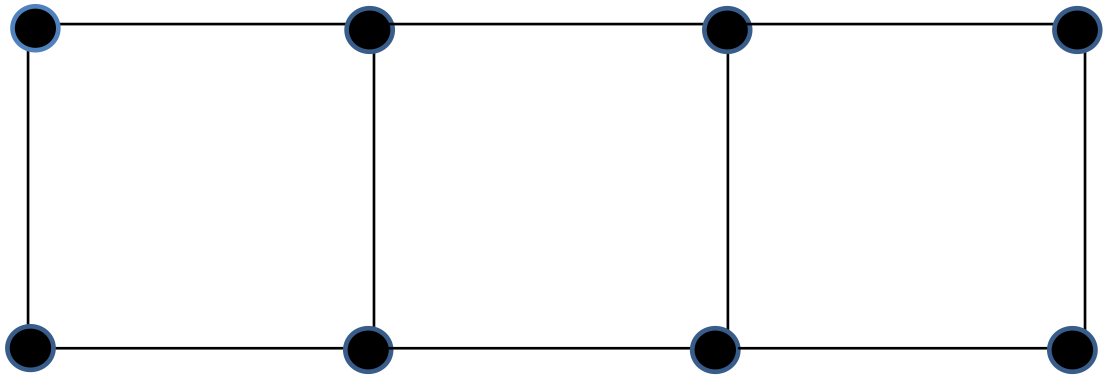
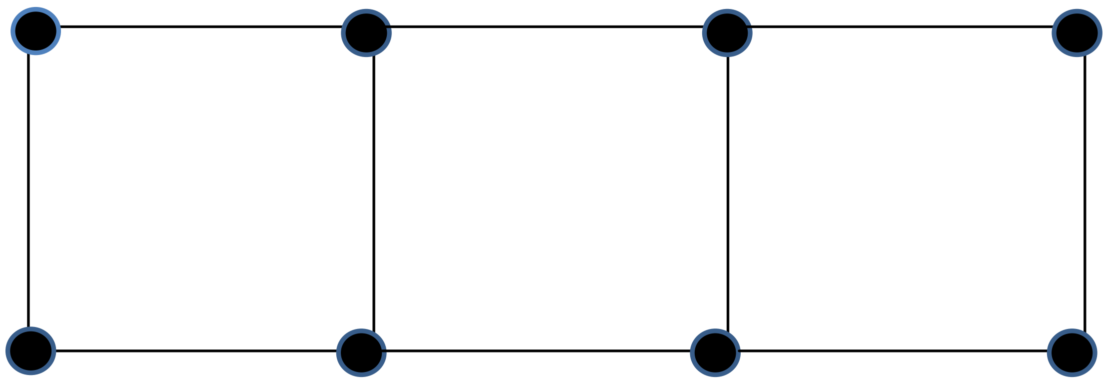
REAL-LIFE PROBLEM

- Security guarding a city
  - Would want to use the least number of guards to see all possible areas



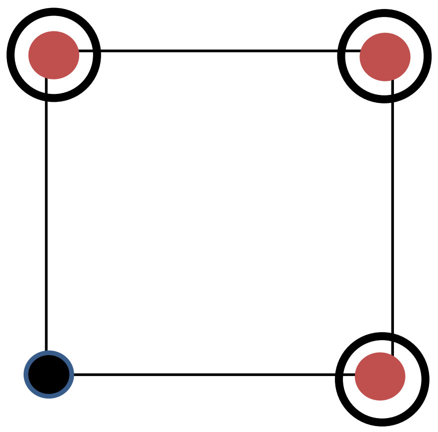
REAL-LIFE PROBLEM

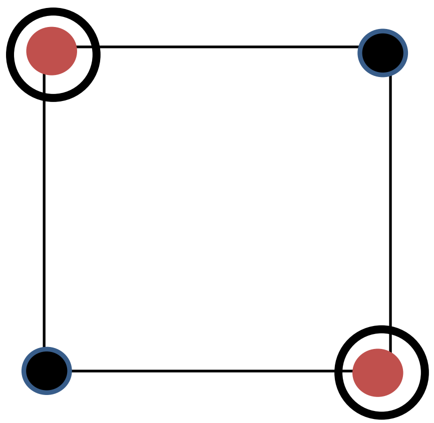
- This problem is a real-life example of minimum dominating sets



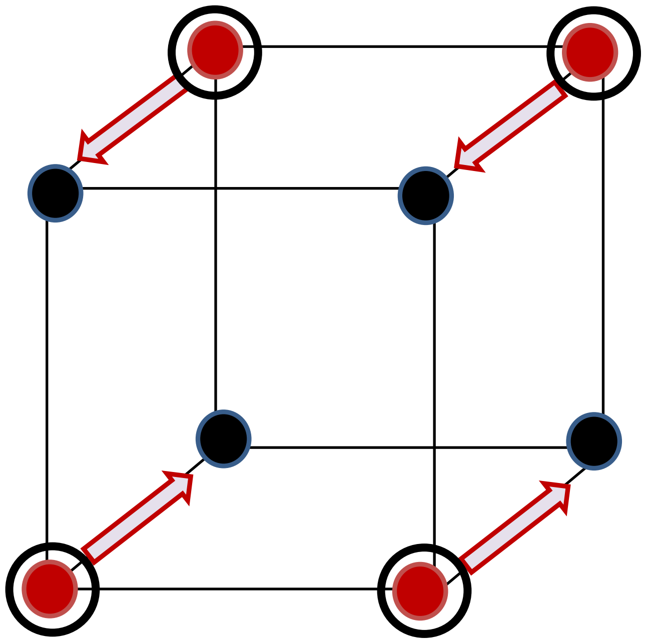
DEFINITIONS

- **Dominating set**: a set of vertices in a graph with the property that every vertex in the graph is covered, meaning it is either in the set or is adjacent to a vertex in the set
- **Minimum dominating set** (γ set): a set whose cardinality is as small as possible
  - The cardinality of such a set is called the domination number of the graph

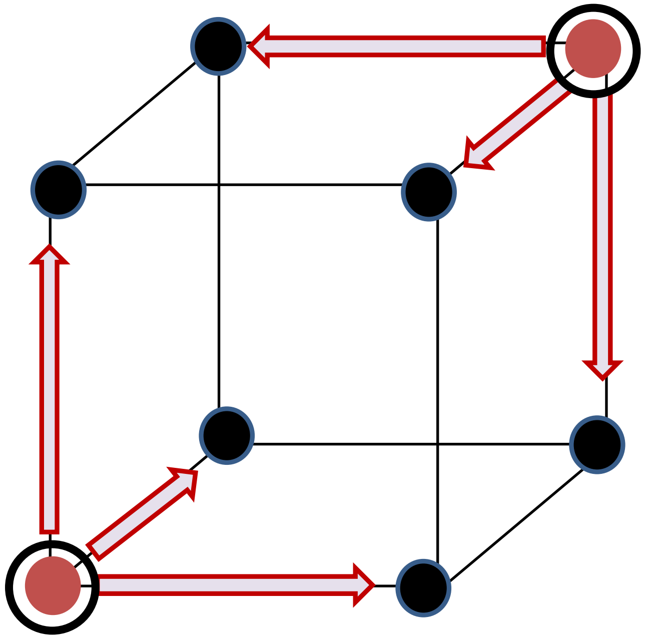




TRANSITION TO 3D



3D Dominating Set Example



3D Minimum Dominating Set Example

GOAL

- Finding minimum number of nodes in minimum dominating sets in terms of 3 dimensions of the grid
- *K. Hutson, S.T. Hedetniemi, and R. Forrester, "Constructing Gamma-Sets of Grids," Journal of Combinatorial Mathematics and Combinatorial Computing, Vol. 95, pp. 3-26, 2015.*

INTEGER PROGRAMMING MODEL

- **Decision Variables:** $x_{ijk} \in \{0,1\}$

- **Objective Function:**

$$\gamma_{l,m,n} = \min \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} x_{ijk}$$

- **Constraints:**

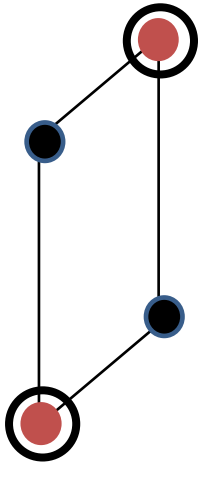
$$x_{(i+1)jk} + x_{(i-1)jk} + x_{i(j+1)k} + x_{i(j-1)k} + x_{ijk} + x_{ij(k-1)} + x_{ij(k+1)} \geq 1 \quad \forall \begin{cases} 1 \leq i \leq l \\ 1 \leq j \leq m \\ 1 \leq k \leq n \end{cases}$$

CHALLENGES

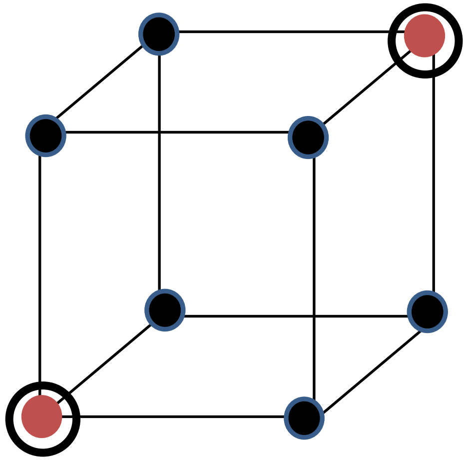
- Computational limitations
  - Computers can only reasonably calculate a certain amount
- NP-hard problem
  - Using results from IP problems to make conjectures

PATTERNS

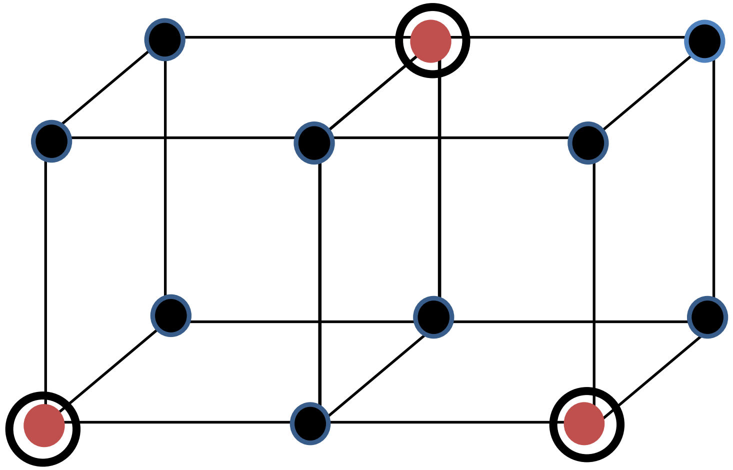
- Found patterns between cardinalities of minimum dominating sets
- 2x2x*n* example



2x2x1 - 2



2x2x2 - 2



2x2x3 - 3

PATTERN CONJECTURES

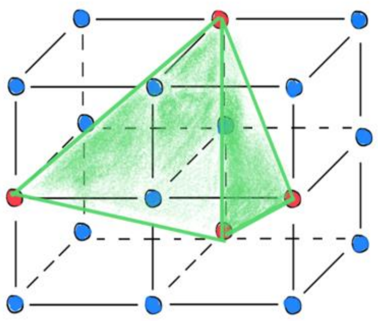
- $2 \times 2 \times n$ series: $\gamma_{2,2,n} = \begin{cases} 2 & \text{if } n = 1 \\ n & \text{otherwise} \end{cases}$
- $2 \times 5 \times n$ series: $\gamma_{2,5,n} = 2n \qquad \forall\, n \geq 17$

PATTERN CONJECTURES CONTINUED

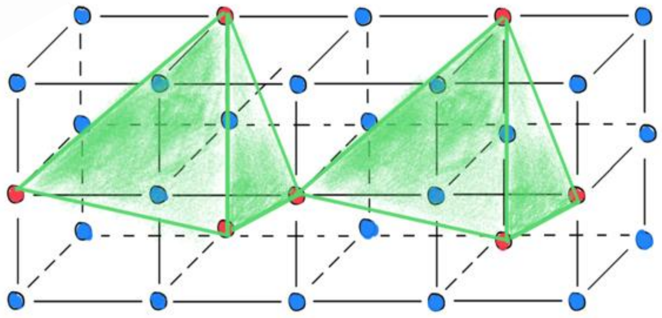
- $2 \times 3 \times n$ series: $\gamma_{2,3,n} = \left\lceil \frac{5n+3}{3} \right\rceil \quad \forall\, n \geq 13$
- $2 \times 4 \times n$ series: $\gamma_{2,4,n} = \begin{cases} \left\lfloor \frac{5n+3}{4} \right\rfloor & n \equiv 1, 3 (\text{mod} 6) \\ \left\lceil \frac{5n+3}{4} \right\rceil & \text{otherwise} \end{cases}$
- $2 \times 6 \times n$ series: $\gamma_{2,6,n} = \begin{cases} \left\lceil \frac{17n+13}{7} \right\rceil & n \equiv 0 (\text{mod} 7) \\ \left\lceil \frac{17n+6}{7} \right\rceil & \text{otherwise} \end{cases}$
- $2 \times 7 \times n$ series: $\gamma_{2,7,n} = \left\lceil \frac{11n+7}{4} \right\rceil \quad \forall\, n \geq 10$

VISUAL PATTERNS IN 3D

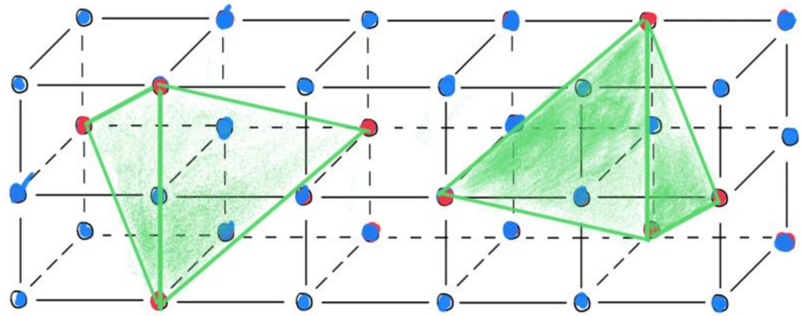
- More difficulty in recognizing visual patterns in 3D graphs
  - More irregular cases observed



$2 \times 3 \times 2$



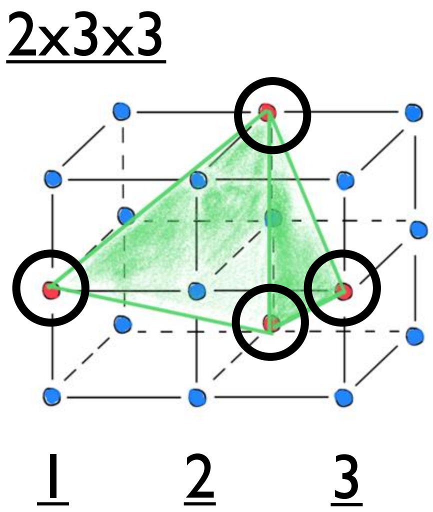
$2 \times 3 \times 5$

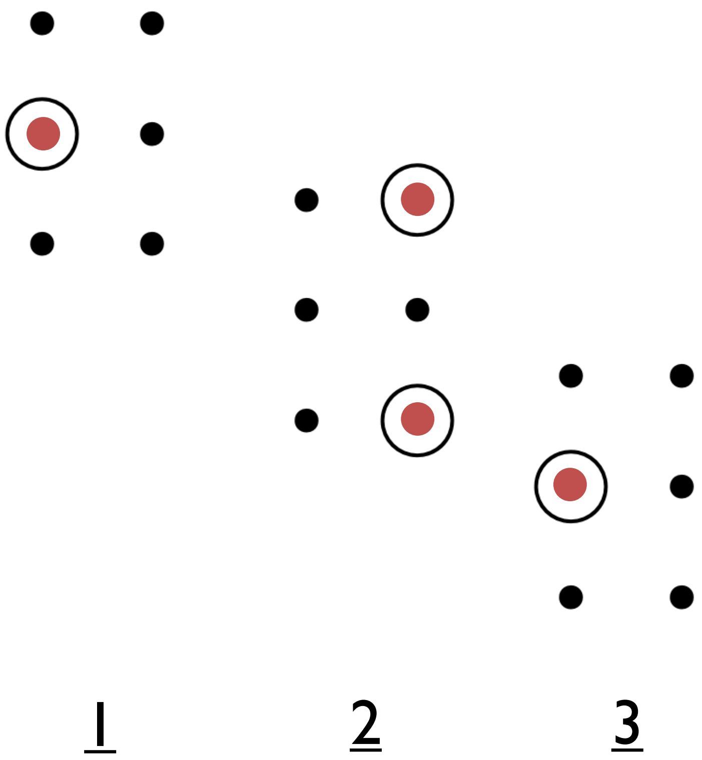


$2 \times 3 \times 6$

STRUCTURAL PATTERN

- Breaking down the 3D grid graph into 2D layers to see how each of them are constructed





COLORED STRUCTURAL PATTERN

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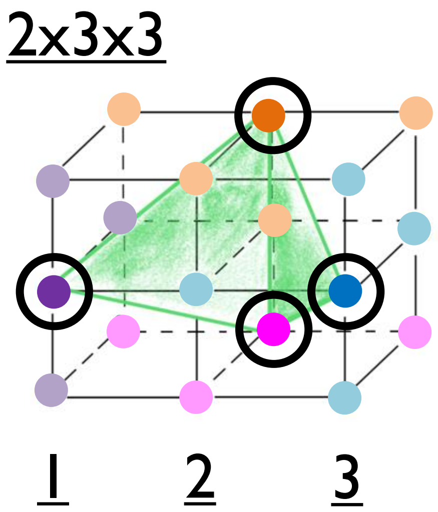
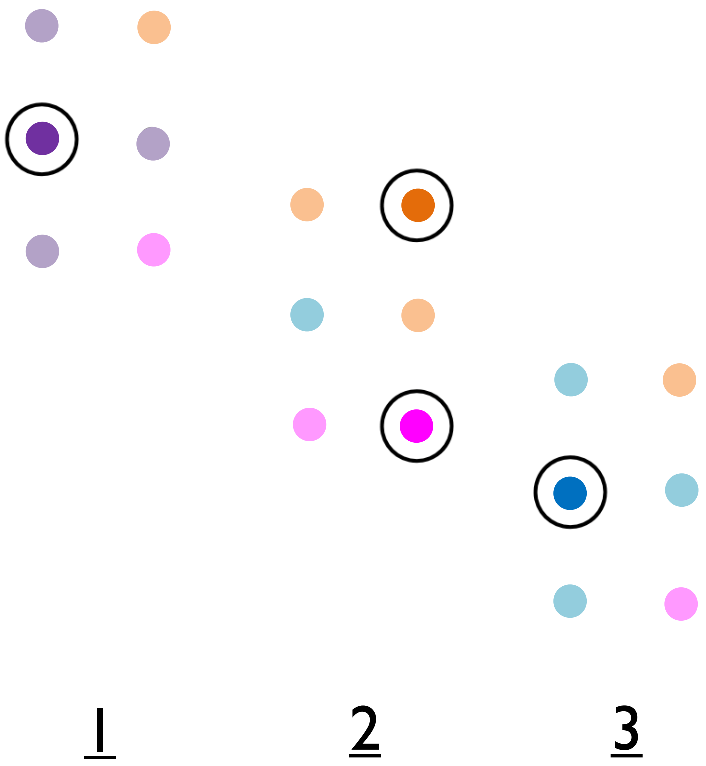
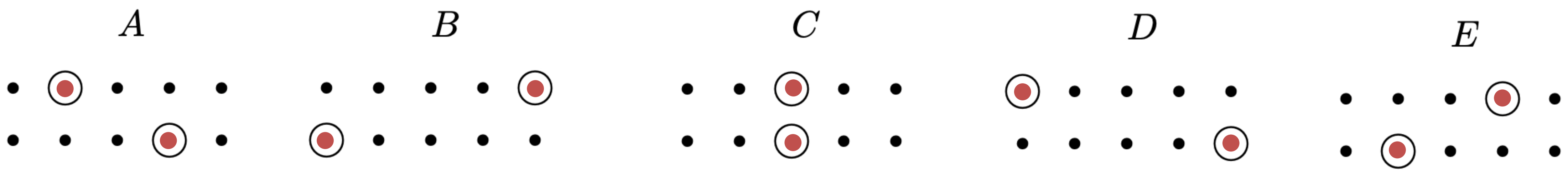




STRUCTURAL PATTERN OF 2X5X*K* SERIES

- Occurrence of "optimal pattern" in the middle with bookends on either side

2x5x*k*



2-2-2-2-2

=> Pattern of 5 layers containing 10 dominators

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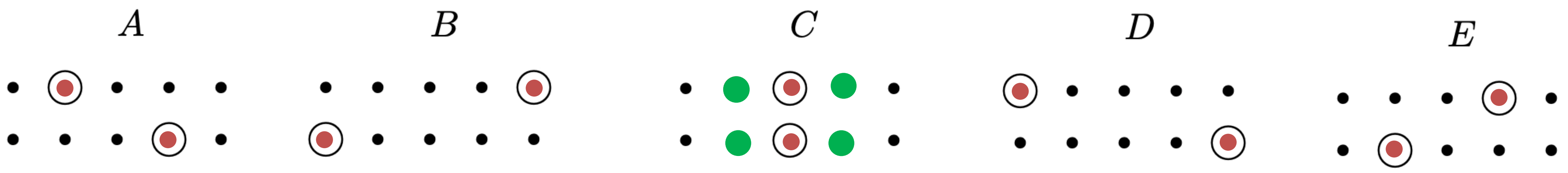

2-2-2-2-2

=> Pattern of 5 layers containing 10 dominators repeated

STRUCTURAL PATTERN OF 2X5X*K* SERIES

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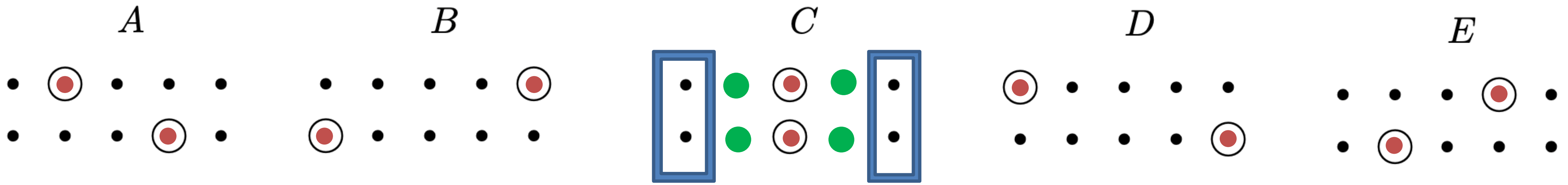

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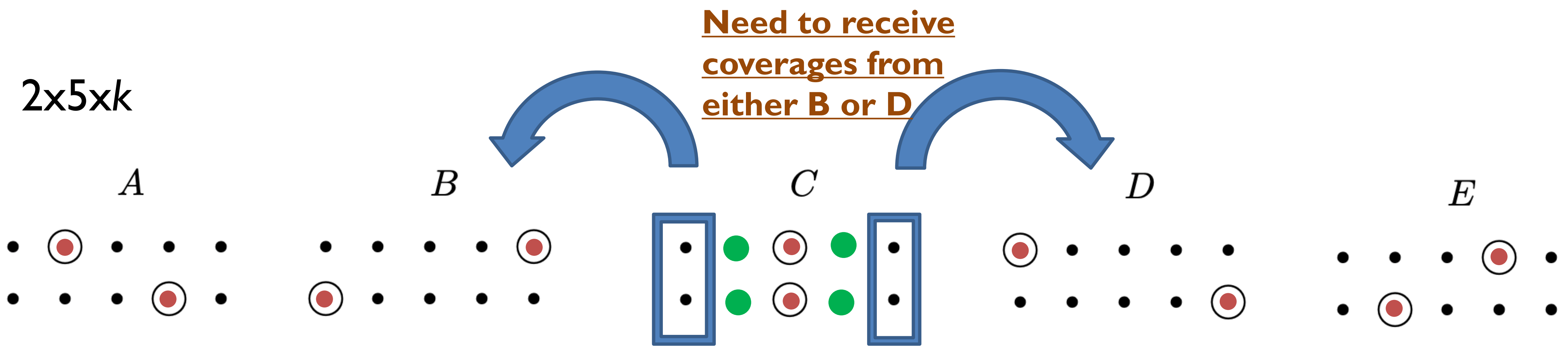

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STRUCTURAL PATTERN

- All vertices in a layer must receive their coverages from either the dominators in its own layer or its adjacent layers

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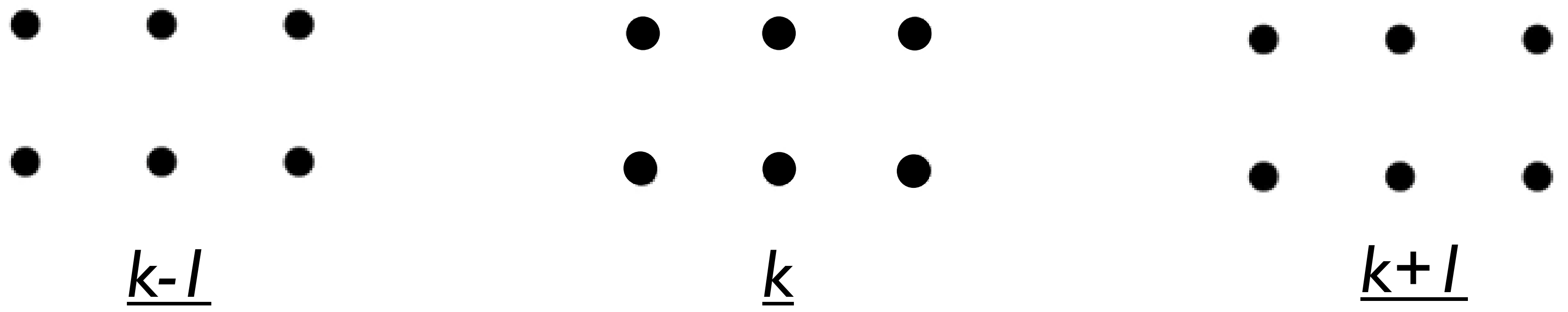
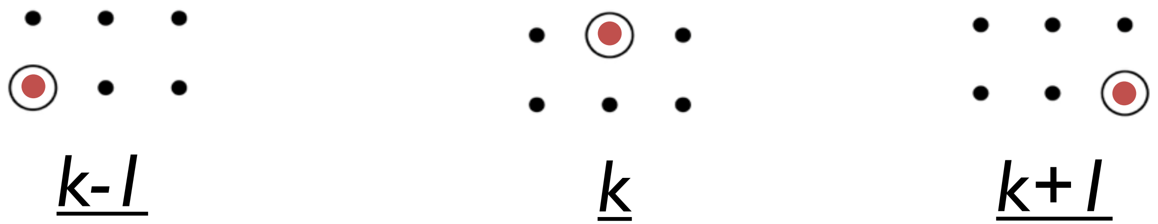
- All vertices in a layer must receive their coverages from either the dominators in its own layer or its adjacent layers
- Layer *k* must be entirely covered by the dominators in layer *k* or layers *k-1* and *k+1*



OPTIMALITY IN CONSTRUCTION

- Optimal arrangements: sum of dominators required is minimal



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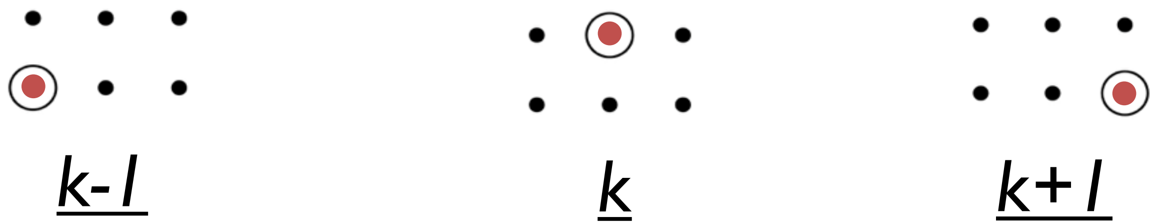

- *k-1* or *k+1* becomes a middle layer

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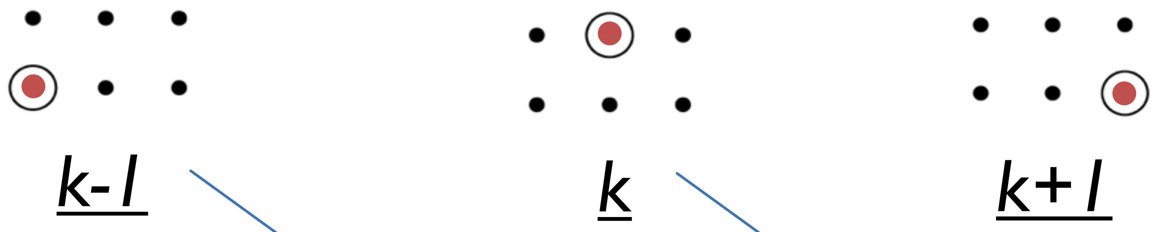

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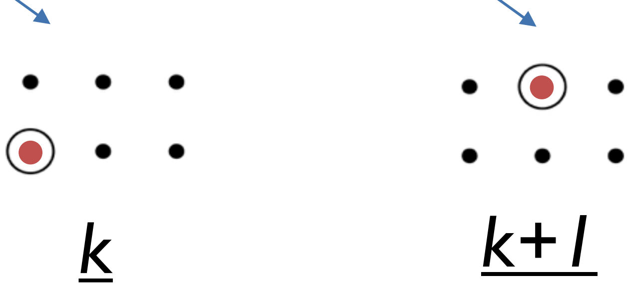

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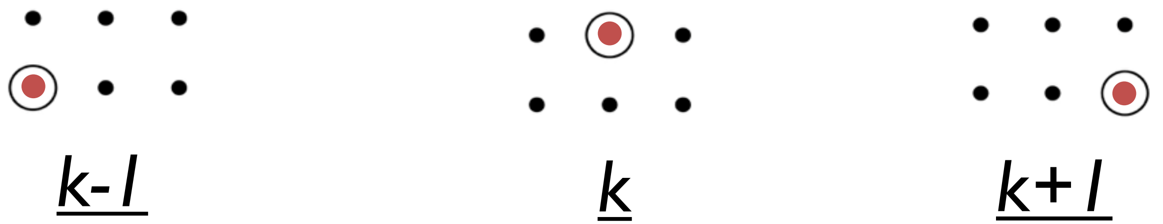

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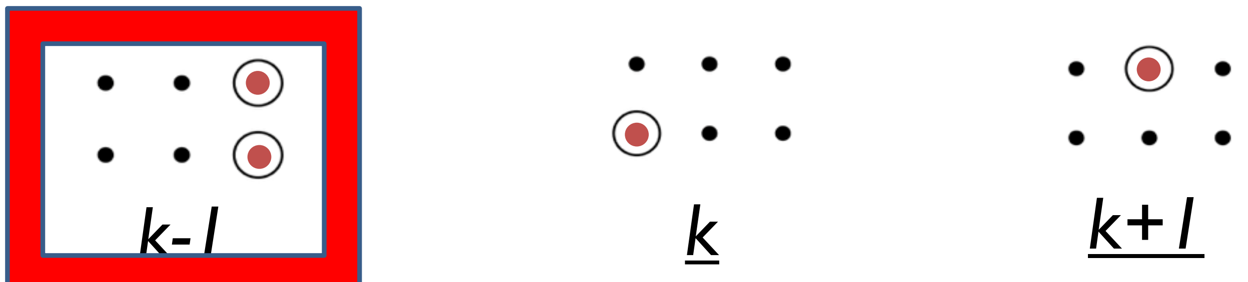

ADDITIONAL RESEARCH

- Continue to find patterns
- Research grids of different dimensions
- Continue proving optimality of the constructions of the dominating set
- Creating encyclopedia of series

ACKNOWLEDGEMENTS

- The other undergraduate students and the professors that collaborated on this project with me
- The Furman Mathematics Department for organization and funding
- The NCUWM conference organizers

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QUESTIONS