COLLEGE OF
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Minimal Base Sizes of Symmetric Groups

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Helpful Definitions

A permutation: is an rearrangement of elements in a set.

Example: permutation (1,2) means 1 → 2 and 2 → 1

(1,7,6)(2,3) means 1 → 7, 7 → 6, 6 → 1, 2 → 3, and 3 → 2

Symmetric Group: the group made up of the permutations on n elements

Example: S_3= ( ), (1,2), (2,3), (1,3), (1,2,3), (1,3,2)

S_n acts on ℓ sets

Example: S_5 acting on subsets of size 3

$\{1,2,3\}^{(2,5)} = \{1,5,3\} = \{1,3,5\}$ Changes the set

$\{1,2,3\}^{(1,3)} = \{3,2,1\} = \{1,2,3\}$ Fixes the set

$\{1,2,3\}^{(4,5)} = \{1,2,3\}$ Fixes the set

$\{1,3,4\}^{(1,2,4)(3,5)} = \{2,5,1\} = \{1,2,5\}$ Changes the set

Example of a Base for 3-sets

Base of a Symmetric group: a set of ℓ-sets in which the only permutation that will fix every ℓ-set in the set is the identity.

Base for S_4: { {1,2,3}, {1,2,4}, {1,3,4} }

permutations that fix the set

{1,2,3}: () (12) (13) (23) (123) (132)

{1,2,4}: () (12) (14) (24) (124) (142)

{1,3,4}: () (13) (14) (34) (134) (142)

Example of a non-base for 3-sets

Non-Base for S_4: { {1,2,3}, {1,2,4}, {1,2,5} }

{1,2,3}: () (12) (13) (23) (123) (132)

{1,2,4}: () (12) (14) (24) (124) (142)

{1,2,5}: () (12) (15) (25) (125) (152)

Summer Research Problem

Base for S_4: { {1,2,3}, {1,2,4}, {1,3,4} }

{1,2,3}: () (12) (13) (23) (123) (132)
{1,2,4}: () (12) (14) (24) (124) (142)
{1,3,4}: () (13) (14) (34) (134) (142)

Find the minimal base size of S_n acting on 2-sets, 3-sets, and 4-sets.

Problem Explained

Example:

- Given the set: {1,2,3,4} subsets of size three include: {1,2,3}, {1,2,4}, {1,3,4}, and {2,3,4}.
- Goal: determine the minimum number of 3-sets we need to form a base.
- Can we do it with two sets?

  No: {{1,2,3} {1,2,4}}, (1,2) fixes both sets

- Can we do it with three sets?

  Yes: {{1,2,3}, {1,2,4}, {1,3,4}}

(n-1) elements are necessary to form a base for S_n

Proof

- Let $\mathfrak{B} = \{B_1, \dots B_k\}$ be a base such that $\{3,4,\dots,(n-1),n\} = \cup\,\mathfrak{B}$.
- Note that $1, 2 \notin \cup\,\mathfrak{B}$
- $B_1^{(1,2)} = B_1$, $B_2^{(1,2)} = B_2 \dots B_k^{(1,2)} = B_k$
- Thus, the permutation (1, 2) fixes every set in $\mathfrak{B}$, meaning that a base cannot be formed with (n-2) elements
- So, you need at least (n-1) elements to form a base.

Lower Bound Formula Example

$\{1,2,3\}$
$\{?,?,?\}$
$\{?,?,?\}$

Case 1: 1, 2, 3 do not exist in any remaining sets then any permutation of {1,2,3} will fix every set.

$\{1,2,3\}$
$\{1,?,?\}$
$\{?,?,?\}$

Case 2: only the first element, 1, exists in remaining sets. Then (2,3) will fix every set.

K=3, ℓ=3

$\{1,2,3\}$
$\{1,4,5\}$
$\{2,4,6\}$

$$3 + \frac{(3)(3-1)}{2} = K + \frac{(K)(\ell-1)}{2} = \frac{K\ell + K}{2}$$

K=3, ℓ=3 $\qquad \frac{(3)(3)+3}{2} = 6$

General Formula for a Lower Bound

A base of K, ℓ-sets can have at most $\frac{K\ell+K}{2}$ elements

Proof

Let $\mathfrak{B} = \{B_1, \ldots B_k\}$ be a base.
Let t be the total number of elements.
Let s be the elements that appear in exactly one B_i
and r the elements that appear in at least two B_i. Note that $r=(t-s)$.
Suppose $a,b \in B_i$ such that $a \notin B_j$ and $b \notin B_j$ for $i \neq j$.
Then (a, b) fixes every set.
So, at most one element of B_i fails to appear in $\cup\, (\mathfrak{B} \setminus B_i)$.
So, we know that $s \leq k$.

Since repeated elements appear at least twice,

$$K\ell \geq s + 2r = s + 2(t - s) = 2t - s \geq 2t\text{-}k$$

$$K\ell + K \geq 2t$$

$$\frac{K\ell + K}{2} \geq t$$

An Upper Bound Rule for Bases

Lemma: *If* $a,b \in B_i$ *then there exists a* B_m *such that* $|B_m \cap \{a,b\}| = 1$.

Proof:

Let $B_i = \{a, b, c_3, \ldots, c_\ell\}$.

Suppose that $a, b \in B_i$ such that $a \in B_m$ if and only if $b \in B_m$.

Then (a, b) fixes every element of $\mathfrak{B}$

So for every a and b in B_i there is a B_m such that $|B_m \cap \{a, b\}| = 1$.

2-sets Upper Bound

A symmetric group such that S_{3m+a} for $a \in \{0,1\}$ requires $2m$ sets to form a base and S_{3m+2} requires $2m+1$ sets to form a base.

S_3 example:

( ) (1,2) (1,3) (2,3) (1,2,3)

{1,2}

{2,3}

Upper Bound for S_7 example:

{1,2}

{2,3}

{3,4}

{4,5}

{5,6}

Best Upper Bound for S_7 example:

{1,2}

{2,3}

{4,5}

{5,6}

Proof for 2-sets: 2m is the min for S_{3m+a} for a∈ {0,1}

- Case 1: a∈ {0,1} can be done with 2*m*
- Suppose 2*m - 1* sets form a base
- Max elements: $\frac{2K+K}{2} = \frac{2(2m-1)+(2m-1)}{2} = \frac{4m-2+2m-1}{2} = \frac{6m-3}{2} = 3m - \frac{3}{2}$
- $3m - \frac{3}{2} < n - 1 \leq (3m+0) - 1 < (3m+1) - 1$
- So you cannot form a base with 2*m – 1* sets

Proof for 2-sets 2m+1 is the minimum for S_{3m+2}

- Case 2: $n = 3m + 2$ can be done with $2m$ +1 sets
- Suppose $2m$ sets form a base
- Max elements: $\frac{2K+K}{2} = \frac{2(2m)+(2m)}{2} = \frac{4m+2m}{2} = \frac{6m}{2} = 3m$
- $3m < n - 1 \leq (3m + 2) - 1$
- So you cannot form a base with $2m$ sets

3-Sets Upper Bound

A symmetric group such that S_{2m+a} for $a \in \{0,1\}$ requires *m* sets to form a base for $m > 2$.

Base example for S_6 and S_7

$\{1, 2, 3\}$
$\{1, 4, 5\}$
$\{2, 4, 6\}$

Base example for S_8 and S_9

$\{1, 2, 3\}$
$\{1, 4, 5\}$
$\{2, 6, 7\}$
$\{4, 6, 8\}$

3-Sets Proof: m is the minimum for S_{2m+a} for a∈ $\{0,1\}$ such that $m > 2$.

- We can do it with *m*
- Suppose *m - 1* sets form a base.
- Max elements: $\frac{3K+K}{2} = \frac{3(m-1)+(m-1)}{2} = \frac{3m-3+m-1}{2} = \frac{4m-4}{2}$ = *2m-2*
- *2m-2*$< n - 1 \leq (2m) - 1 < (2m+1) - 1$
- So you cannot do it with *m – 1* sets

4-sets Upper Bound

S_{5m+a} for a$\in \{-1,0,1\}$ have an upper bound of $2m$ sets

S_{5m+a} such that a$\in \{2,3\}$ have an upper bound of $2m+1$ sets

Base example for S_{10} and S_{11}

{1,2,3,4}

{1,5,6,7}

{2,5,8,9}

{3,6,8,10}

Bases for $S_7, S_8, S_9, S_{10}, S_{11}$, and S_{12}

These bases will form the building blocks for our upper bound formula for 4-sets.

Base S_7	Base S_8	Base S_9	Base S_{10}	Base S_{11}	Base S_{12}
{1,2,3,4}	{1,2,3,4}	{1,2,3,4}	{1,2,3,4}	{1,2,3,4}	{1,2,3,4}
{1,2,5,6}	{1,2,5,6}	{1,5,6,7}	{1,5,6,7}	{1,5,6,7}	{1,5,6,7}
{2,3,5,7}	{2,3,5,8}	{2,5,8,9}	{2,5,8,9}	{2,5,8,9}	{2,6,8,9}
		{3,5,6,8}	{3,6,8,10}	{3,6,8,10}	{3,8,10,11}
					{,5,10,12,13}

Upper Bound 4-set Building Block Idea

Because we already have the bases for $S_{7,}$ $S_{8,}$ $S_{9,}$ S_{10} and S_{12}we are able to use these as building blocks to create bases for every symmetric group.

Base for S_{10} and S_{11}

{1,2,3,4}

{1,5,6,7}

{2,5,8,9}

{3,6,8,10}

Base for S_{20} and S_{21}

{1,2,3,4}.	{11,12,13,14}
{1,5,6,7}	{11,15,16,17}
{2,5,7,9}.	{12,15,18,19}
{3,6,7,10}.	{13,16,18,20}

4-sets Upper Bound

S_{5m+a} for $a \in \{-1, 0, 1\}$ have an upper bound of $2m$ sets

S_{5m+a} such that $a \in \{2, 3\}$ have an upper bound of $2m+1$ sets

4-Sets Proof :2m is the min S_{5m+a} for a$\in \{-1,0,1\}$

- Case 1: Let $a \in \{-1, 0, 1\}$. Suppose $2m-1$ sets form a base.
- Max number of elements:
- $\frac{4K+K}{2} = \frac{4(2m-1)+(2m-1)}{2} = \frac{8m-4+2m-1}{2} = \frac{10m-5}{2} = 5m - \frac{5}{2}$
- $5m - \frac{5}{2} < n - 1 \leq (5m-1) - 1 < (5m) - 1 < (5m+1) - 1$
- *2m - 1* sets are not enough to form a base.

4-Sets Proof :$2m+1$ is the min S_{5m+a} for a∈ {2,3}

- Case 2: Let $a \in \{2,3\}$. Suppose $2m$ sets form a base.
- $\frac{4(2m)+(2m)}{2} = \frac{10m}{2} = 5m$
- $5m < n - 1 \leq (5m + 2) - 1 < (5m + 3) - 1$
- $2m$ sets are not enough to form a base.

Conclusion

- Found minimal base size of S_n acting on 2-sets, 3-sets, and 4-sets.

Questions?