Minimal Base Sizes of Symmetric Groups

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Helpful Definitions

A permutation: is an rearrangement of elements in a set.

Example: permutation (1,2) means 1 → 2 and 2 → 1
(1,7,6)(2,3) means 1 → 7, 7 → 6, 6 → 1, 2 → 3, and 3 → 2

Symmetric Group: the group made up of the permutations on n elements

Example: $S_3 = ( ), (1,2), (2,3), (1,3), (1,2,3), (1,3,2)$
\( S_n \) acts on \( \ell \) sets

**Example:** \( S_5 \) acting on subsets of size 3

\[
\begin{align*}
\{1,2,3\}^{(2,5)} &= \{1,5,3\} = \{1,3,5\} & \text{Changes the set} \\
\{1,2,3\}^{(1,3)} &= \{3,2,1\} = \{1,2,3\} & \text{Fixes the set} \\
\{1,2,3\}^{(4,5)} &= \{1,2,3\} & \text{Fixes the set} \\
\{1,3,4\}^{(1,2,4)(3,5)} &= \{2,5,1\} = \{1,2,5\} & \text{Changes the set}
\end{align*}
\]
Example of a Base for 3-sets

**Base of a Symmetric group:** a set of $\ell$-sets in which the only permutation that will fix every $\ell$-set in the set is the identity.

Base for $S_4$: \{ \{1,2,3\}, \{1,2,4\}, \{1,3,4\} \}

permutations that fix the set

\{1,2,3\}: () (12) (13) (23) (123) (132)
\{1,2,4\}: () (12) (14) (24) (124) (142)
\{1,3,4\}: () (13) (14) (34) (134) (142)
Example of a non-base for 3-sets

Non-Base for $S_4$: \{ \{1,2,3\}, \{1,2,4\}, \{1,2,5\} \}

\{1,2,3\}: () (12) (13) (23) (123) (132)
\{1,2,4\}: () (12) (14) (24) (124) (142)
\{1,2,5\}: () (12) (15) (25) (125) (152)
Summer Research Problem

Base for $S_4$: $\{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}\}$

$\{1,2,3\}$: $()$ (12) (13) (23) (123) (132)
$\{1,2,4\}$: $()$ (12) (14) (24) (124) (142)
$\{1,3,4\}$: $()$ (13) (14) (34) (134) (142)

Find the minimal base size of $S_n$ acting on 2-sets, 3-sets, and 4-sets.
Problem Explained

Example:

• Given the set: \{1,2,3,4\} subsets of size three include: \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, and \{2,3,4\}.

• Goal: determine the minimum number of 3-sets we need to form a base.

• Can we do it with two sets?
  No: \{{{1,2,3} {1,2,4}}}, (1,2) fixes both sets

• Can we do it with three sets?
  Yes: \{{{1,2,3}, {1,2,4}, {1,3,4}}


(n-1) elements are necessary to form a base for $S_n$

Proof

• Let $\mathcal{B} = \{B_1, \ldots, B_k\}$ be a base such that \{3,4, ... , (n − 1), n\} = $\cup \mathcal{B}$.
• Note that 1, 2 \notin $\cup \mathcal{B}$
• $B_1^{(1,2)} = B_1$, $B_2^{(1,2)} = B_2$ ... $B_k^{(1,2)} = B_k$
• Thus, the permutation (1, 2) fixes every set in $\mathcal{B}$, meaning that a base cannot be formed with (n-2) elements
• So, you need at least (n-1) elements to form a base.
Lower Bound Formula Example

Case 1: 1, 2, 3 do not exist in any remaining sets then any permutation of \{1,2,3\} will fix every set.

Case 2: only the first element, 1, exists in remaining sets. Then \{1,4,5\} \{2,4,6\}, (2,3) will fix every set.

\[
3 + \frac{(3)(3-1)}{2} = K + \frac{(K)(\ell-1)}{2} = \frac{K\ell+K}{2}
\]

\[
K=3, \ \ell=3 \quad \frac{(3)(3)+3}{2} = 6
\]
General Formula for a Lower Bound

A base of $K$, $\ell$-sets can have at most $\frac{K \ell + K}{2}$ elements.
Proof

Let $\mathcal{B} = \{ B_1, \ldots B_k \}$ be a base.
Let $t$ be the total number of elements.
Let $s$ be the elements that appear in exactly one $B_i$
and $r$ the elements that appear in at least two $B_i$. Note that $r = (t - s)$.
Suppose $a, b \in B_i$ such that $a \notin B_j$ and $b \notin B_j$ for $i \neq j$.
Then $(a, b)$ fixes every set.
So, at most one element of $B_i$ fails to appear in $\bigcup (\mathcal{B} \setminus B_i)$.
So, we know that $s \leq k$.

Since repeated elements appear at least twice,
\[
K\ell \geq s + 2r = s + 2(t - s) = 2t - s \geq 2t - k
\]
\[
K\ell + K \geq 2t
\]
\[
\frac{K\ell + K}{2} \geq t
\]
An Upper Bound Rule for Bases

Lemma: If \( a, b \in B_i \) then there exists a \( B_m \) such that \( |B_m \cap \{a, b\}| = 1 \).

Proof:
Let \( B_i = \{a, b, c_3, \ldots, c_\ell\} \).
Suppose that \( a, b \in B_i \) such that \( a \in B_m \) if and only if \( b \in B_m \).
Then \((a, b)\) fixes every element of \( \mathfrak{B} \).
So for every \( a \) and \( b \) in \( B_i \) there is a \( B_m \) such that \( |B_m \cap \{a, b\}| = 1 \).
2-sets Upper Bound

A symmetric group such that $S_{3m+a}$ for $a \in \{0,1\}$ requires $2m$ sets to form a base and $S_{3m+2}$ requires $2m+1$ sets to form a base.

$S_3$ example:
( ) (1,2) (1,3) (2,3) (1,2,3)
{1,2}
{2,3}

Upper Bound for $S_7$ example:

{1,2}
{2,3}
{3,4}
{4,5}
{5,6}

Best Upper Bound for $S_7$ example:

{1,2}
{2,3}
{4,5}
{5,6}
Proof for 2-sets: 2m is the min for $S_{3m+a}$ for $a \in \{0, 1\}$

- Case 1: $a \in \{0, 1\}$ can be done with $2m$
- Suppose $2m - 1$ sets form a base
- Max elements:
  \[
  \frac{2K + K}{2} = \frac{2(2m-1) + (2m-1)}{2} = \frac{4m-2+2m-1}{2} = \frac{6m-3}{2} = 3m - \frac{3}{2}
  \]
  
- $3m - \frac{3}{2} < n - 1 \leq (3m + 0) - 1 < (3m + 1) - 1$
- So you cannot form a base with $2m - 1$ sets
Proof for 2-sets $2m+1$ is the minimum for $S_{3m+2}$

- Case 2: $n = 3m + 2$ can be done with $2m + 1$ sets
- Suppose $2m$ sets form a base
- Max elements: $\frac{2K + K}{2} = \frac{2(2m) + (2m)}{2} = \frac{4m + 2m}{2} = \frac{6m}{2} = 3m$
- $3m < n - 1 \leq (3m + 2) - 1$
- So you cannot form a base with $2m$ sets
3-Sets Upper Bound

A symmetric group such that $S_{2m+a}$ for $a \in \{0,1\}$ requires $m$ sets to form a base for $m > 2$.

Base example for $S_6$ and $S_7$  
Base example for $S_8$ and $S_9$

$$\{1, 2, 3\}$$  
$$\{1, 4, 5\}$$  
$$\{2, 4, 6\}$$  

$$\{1, 2, 3\}$$  
$$\{1, 4, 5\}$$  
$$\{2, 6, 7\}$$  
$$\{4, 6, 8\}$$
3-Sets Proof: $m$ is the minimum for $S_{2m+a}$ for $a \in \{0,1\}$ such that $m > 2$.

- We can do it with $m$
- Suppose $m - 1$ sets form a base.
- Max elements: \( \frac{3K+K}{2} = \frac{3(m-1)+(m-1)}{2} = \frac{3m-3+m-1}{2} = \frac{4m-4}{2} = 2m-2 \)
  - $2m-2 < n - 1 \leq (2m) - 1 < (2m + 1) - 1$
- So you cannot do it with $m - 1$ sets
4-sets Upper Bound

$S_{5m+a}$ for $a \in \{-1,0,1\}$ have an upper bound of $2m$ sets

$S_{5m+a}$ such that $a \in \{2,3\}$ have an upper bound of $2m+1$ sets

Base example for $S_{10}$ and $S_{11}$

\[ \{1,2,3,4\} \]
\[ \{1,5,6,7\} \]
\[ \{2,5,8,9\} \]
\[ \{3,6,8,10\} \]
Bases for $S_7, S_8, S_9, S_{10}, S_{11}$, and $S_{12}$

These bases will form the building blocks for our upper bound formula for 4-sets.

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<th>Base $S_7$</th>
<th>Base $S_8$</th>
<th>Base $S_9$</th>
<th>Base $S_{10}$</th>
<th>Base $S_{11}$</th>
<th>Base $S_{12}$</th>
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Upper Bound 4-set Building Block Idea

Because we already have the bases for $S_7, S_8, S_9, S_{10}$ and $S_{12}$ we are able to use these as building blocks to create bases for every symmetric group.

Base for $S_{10}$ and $S_{11}$

$\{1,2,3,4\}$
$\{1,5,6,7\}$
$\{2,5,8,9\}$
$\{3,6,8,10\}$

Base for $S_{20}$ and $S_{21}$

$\{1,2,3,4\}$. $\{11,12,13,14\}$
$\{1,5,6,7\}$ $\{11,15,16,17\}$
$\{2,5,7,9\}$. $\{12,15,18,19\}$
$\{3,6,7,10\}$. $\{13,16,18,20\}$
4-sets Upper Bound

$S_{5m+a}$ for $a \in \{-1,0,1\}$ have an upper bound of $2m$ sets

$S_{5m+a}$ such that $a \in \{2,3\}$ have an upper bound of $2m+1$ sets
4-Sets Proof: $2m$ is the min $S_{5m+a}$ for $a \in \{-1,0,1\}$

- Case 1: Let $a \in \{-1,0,1\}$. Suppose $2m-1$ sets form a base.
- Max number of elements:
  \[
  \frac{4K+K}{2} = \frac{4(2m-1)+(2m-1)}{2} = \frac{8m-4+2m-1}{2} = \frac{10m-5}{2} = 5m - \frac{5}{2}
  \]
  \[
  5m - \frac{5}{2} < n - 1 \leq (5m - 1) - 1 < (5m) - 1 < (5m + 1) - 1
  \]
- $2m - 1$ sets are not enough to form a base.
4-Sets Proof: $2m + 1$ is the min $S_{5m+a}$ for $a \in \{2,3\}$

- Case 2: Let $a \in \{2,3\}$. Suppose $2m$ sets form a base.
  - $\frac{4(2m)+(2m)}{2} = \frac{10m}{2} = 5m$
  - $5m < n - 1 \leq (5m + 2) - 1 < (5m + 3) - 1$
  - $2m$ sets are not enough to form a base.
Conclusion

• Found minimal base size of $S_n$ acting on 2-sets, 3-sets, and 4-sets.
Questions?