Using the Crushtaceans of Fully Augmented Links to Investigate Cheeger Constants

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Crushtaceans

From a link diagram, we will place an unknotted component around each twist region. Then we remove all full twists.

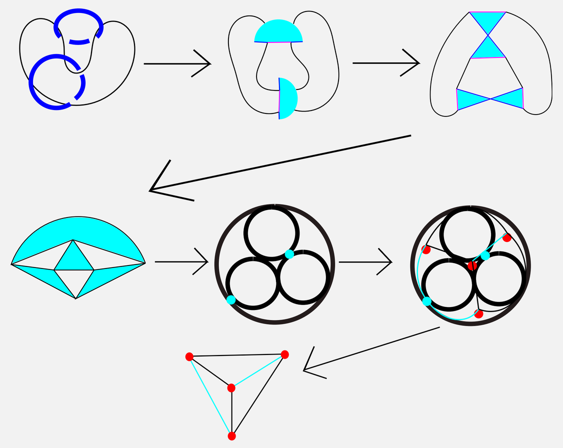

Figure: Taking the FAL of the B-Rings to the Crushtacean

The resulting manifold is hyperbolic [1].

Pretty Edge Cuts

We will can define something in the Crushtacean analogous to surfaces that disconnect the complement of an FAL.

Definition

Let $G = \{V, E\}$ be a trivalent graph and G_1 and G_2 connected subgraphs such that their vertices partition V each has more than one vertex. An edge cut consisting of all edges between some $v_1 \in V_1$ and some $v_2 \in V_2$ is a *pretty edge cut*.

Definition

A *perfect matching* is some set of paired vertices that matches each vertex to exactly one other vertex connected via an edge. A *painting* of a graph G is all edges between matched vertices in a particular perfect matching on G.

Theorem

Consider the trivalent graph G with vertices V, edges E, and some painting. If we take some k-pretty edge cut on G, the parity of the painted edges in the k-edge cut is the same as the parity of k.

Idea of proof: We must have an even number of vertices in a trivalent graph. After we split our graph G into two, we form trivalent graphs out of our subgraphs and find the parity of our subgraphs. This forces the parity of painted edges.

Triangle Vertex Expansions

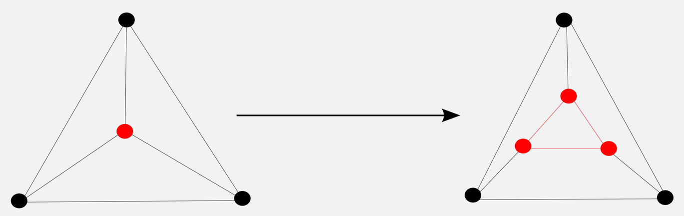

Figure: **Triangle Vertex Expansion (TVE):** When we perform a triangle vertex expansion, the red vertex is replaced with the red vertices and edges on the right.

Cheeger Constants

Definition

The *Cheeger Constant of some graph* $G = \{V, E\}$ is the minimal

$$\frac{k}{\min\{Vol(A), Vol(B)\}}$$

where a k pretty edge cut separates A and B and $Vol(U)$ is the sum of degrees for vertices in U.

Results about TVEs

Theorem

The graph resulting from a TVE of a trivalent graph G has a Cheeger constant no greater than that of G.

Idea of proof: We never get rid of an edge cut through a TVE, and only increase the volumes. The proof of this also creates the concept of "analogous cuts" from G in its TVE.

Theorem

For any graph $G = \{V, E\} \in B_4$, the upper bound of the Cheeger Constant of G is

$$\frac{4}{\frac{3}{2}|V|}.$$

Idea of proof: The 4 cut described in the construction of this set realizes this fraction. This is an upper bound, not an equality, because after a certain number of TVEs, a 3 cut realizes a smaller Cheeger Constant.

Proposition

For a 3-edge connected trivalent graph G together with a TVE on G that produces some G', we find G' is also 3-edge connected.

Idea of proof: We can use the same types of paths in G' as we did in G, with a few modifications for certain cases.

Theorem

The Cheeger constant of any $G = \{V, E\} \in B_3$ will be

$$\frac{2}{|V|}.$$

Idea of Proof: In this set of graphs, there is a 3 edge cut that divides the volume in half. We then use the above proposition to show that we have minimized the numerator and maximized the denominator, which will necessarily create the minimal fraction.

Future Work

1. Find upper or lower bounds on the Cheeger Constant of the FAL in terms of the Crushtacean.
2. Given some Crushtacean, determine the type of surface that produces cut that realizes the Cheeger Constant in the FAL.
3. Determine the behaviors of Crushtaceans when we allow graphs not generated from K_4.
4. Define other families of Crushtaceans with a certain formula for or a bound on their Cheeger Constant.

B_3 and B_4

Notation

Let B_4 contain the following type of graphs: In K_4, there exists a 4 edge cut that divides the volume in half. Performing TVEs in pairs, one on either side of the edge cut.

Notation

Let B_3 contain the following type of graphs: Perform a TVE on one vertex in K_4. Expand the same number of times in the new vertices and the old vertices each section of this graph.

Cheeger Constants of Crushtaceans and FALs

Each k pretty edge cut with n painted edges on a Crushtacean corresponds to a non-trivial cut in the manifold consisting of a k punctured sphere together with n crossing disks. Then the cutting surface has area $n(2\pi) + 2(k-2)\pi$. Using results from [2], we can bound the Cheeger Constants of FALs corresponding to graphs in B_4. If our 4 edge cut has n painted edges, and we have performed t pairs of TVEs, then this upper bound is $\frac{4\pi + n(2\pi)}{\frac{1}{2}(4t+2)v_8} = \frac{4\pi}{(2t+1)v_8}$.

References

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