Exploring the Min-Plus and Max-Plus Finite Tropical Semirings

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MIN-PLUS

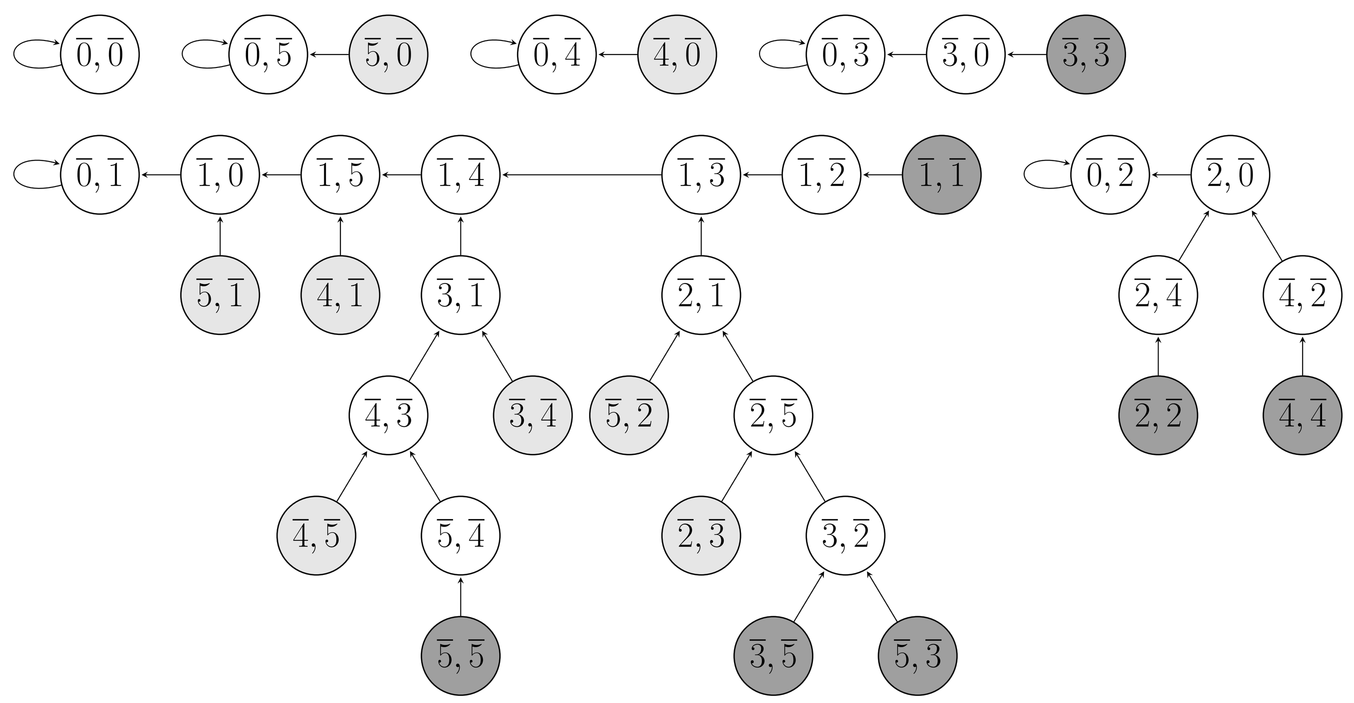

Figure 1: $\Psi(S_6), \overline{x} \oplus \overline{y} = \min(\overline{x}, \overline{y})$

Theorem (On Sources)

Let $(\overline{x}, \overline{y}) \in \Psi(S_n)$ be a source. Then:

- *x and y satisfy :* $\begin{cases} 2x > y & x < y \\ 2x - n > y & x > y \\ x, y > 0 & x = y \end{cases}$
- *$(\overline{x}, \overline{y})$ where $x \leq y$ is a true source if $x = y$ or $\frac{x+n}{2} < y < 2x$.*
- *If $(\overline{x}, \overline{y})$ is a true source, then $(\overline{x}, \overline{y})$ is found in a connected component of $\Psi(S_n)$ terminating in $(\overline{0}, \overline{a})$ where $0 < a \leq \frac{n}{2}$.*

Theorem (On Connected Components)

Let $(\overline{0}, \overline{a})$ be the terminal vertex of a connected component of $\Psi(S_n)$. Then:

- *If a is a factor of n, the connected component will contain $(\overline{a}, \overline{a})$ as a source.*
- *If $0 < a < \frac{n}{2}$, the connected component will contain the fundamental path*

$$(\overline{a}, \overline{n - xa}) \to \ldots \to (\overline{a}, \overline{n - 3a}) \to (\overline{a}, \overline{n - 2a}) \to (\overline{a}, \overline{n - a}) \to (\overline{a}, \overline{0}) \to (\overline{0}, \overline{a})$$

  where $(\overline{a}, \overline{n - xa})$ is a source.
- *If the connected component contains the fundamental path, a smaller value of $\overline{a}$ results in a longer fundamental path.*
- *If $a = 1$, the connected component contains the longest fundamental path of $\Psi(S_n)$.*

Theorem (On the Diameter)

Let $(\overline{0}, \overline{a})$ be the terminal vertex of a connected component of $\Psi(S_n)$. Then:

- *If $\frac{n}{2} < a \leq n - 1$, the connected component will have a diameter of 1.*
- *If n is even and $a = \frac{n}{2}$, the connected component will have a diameter of 2.*
- *If $0 < a < \frac{n}{2}$, the connected component will have a diameter of at least 2.*
- *If $a = 1$, the diameter of the connected component is the diameter of $\Psi(S_n)$.*

Read our work!

Key Terms

- **Tropical Arithmetic** A form of arithmetic in which addition ($\oplus$) is defined as the minimum or maximum of two values and multiplication ($\otimes$) is defined as the sum of two values.
- **Tropical Semiring** A set with binary operations of tropical addition and multiplication that are each commutative, associative, distributive, and have an identity. Note that a semiring does not contain additive inverses.
- **Finite Tropical Semiring** A tropical semiring where $S_n := \mathbb{Z}_n \cup \{\infty\}$ or $S_n := \mathbb{Z}_n \cup \{-\infty\}$. Integers $z \in \mathbb{Z}$ are represented in S_n by their equivalence class $\overline{x}$ where $x \equiv z \mod n$.
- **Digraph of the Finite Tropical Semiring** A graph containing vertices and connecting arrows. The vertices consist of two elements $\overline{x}, \overline{y}$ from S_n, and $(\overline{x}, \overline{y}) \to (\overline{x} \oplus \overline{y}, \overline{x} \otimes \overline{y})$.
- **Connected Component** A connected component of $\Psi(S_n)$ consists of a maximal network of all connected vertices.

Why Digraphs?

- Semirings include two binary operations, and traditional Cayley tables only describe one of the operations while digraphs describe both. [1]
- Digraphs provide a single visualization of the binary operations. [1]
- Unlike other alternatives, the elegance of digraphs provides a simple demonstration of the structure of a semiring. [2]

Infinite Elements

The connected components containing vertices with infinite elements were congruent for all $\Psi(S_n)$ where $0 \leq a < n$. The left connected components exist in the min-plus digraphs and the right connected components exist in the max-plus digraphs.

Significant Joint Results

- **Nonisomorphic** The min-plus and max-plus finite tropical semirings are nonisomorphic.
- **In-degree of 1** Only vertices of the form $(\overline{x}, \overline{x}^2) \in \Psi(S_n)$ have an in-degree of 1, and $(\overline{x}, \overline{x}) \to (\overline{x}, \overline{x}^2)$.
- **Source Count** The number of sources in a given $\Psi(S_n)$ is $\sum_{i=1}^{n-1} i$ for $n > 1$.
- **GCD of Vertices** For all vertices $(\overline{x_1}, \overline{y_1})$ and $(\overline{x_2}, \overline{y_2})$ in a connected component of $\Psi(S_n)$, $\gcd(x_1, y_1, n) = \gcd(x_2, y_2, n)$.
- **Subsemirings** Every vertex consisting of elements from a given subsemiring $\langle a \rangle$ are in the union of the connected components containing $(\overline{0}, \overline{c})$ where c is an element of $\langle a \rangle$.
- **Similar Paths** The vertices $(\overline{x}, \overline{y})$ in the cycle of a given connected component in the max-plus digraph where $\gcd(x, y, n) = a$ are the multiplicative inverse of the vertices in the fundamental path of a connected component of the min-plus digraph terminating in $(\overline{0}, \overline{a})$.

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MAX-PLUS

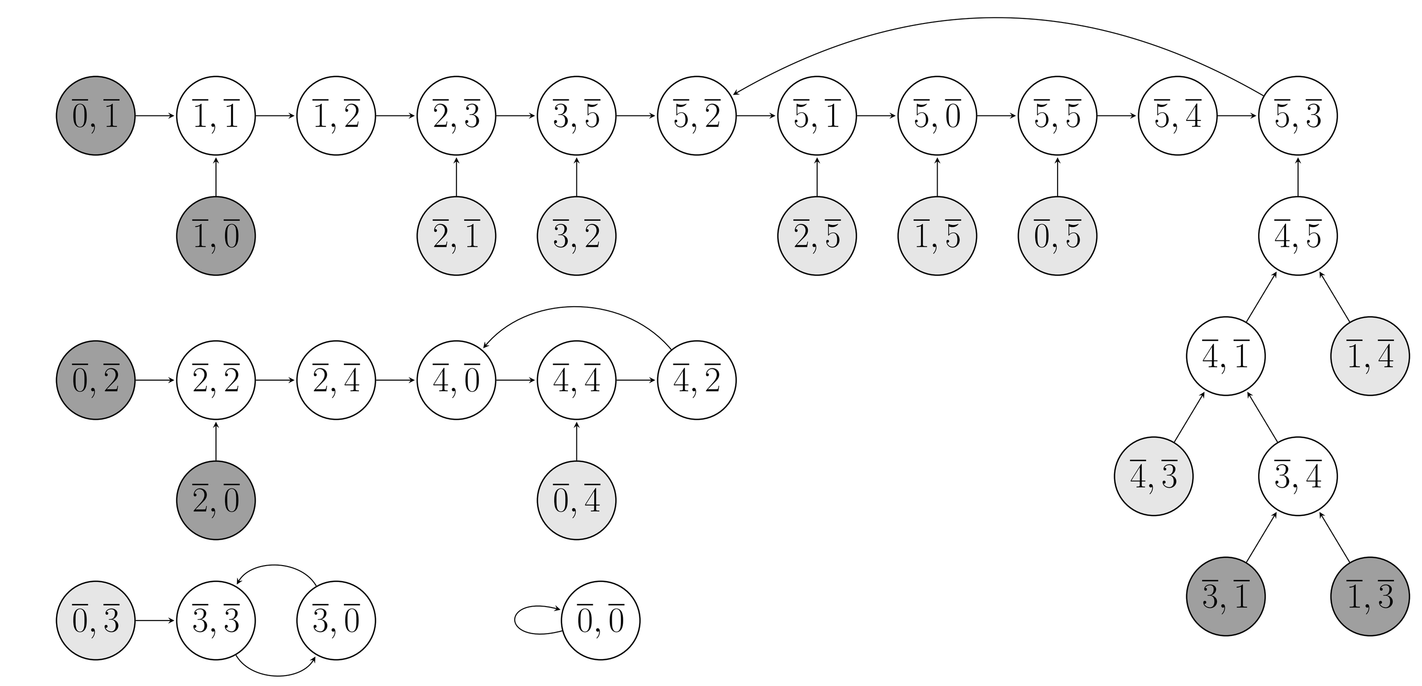

Figure 2: $\Psi(S_6), \overline{x} \oplus \overline{y} = \max(\overline{x}, \overline{y})$

Theorem (On Sources)

Let $(\overline{x}, \overline{y}) \in \Psi(S_n)$ be a source. Then:

- *x and y satisfy :* $\begin{cases} 2x < y & x < y \\ 2x - n < y & x > y \end{cases}$
- *$(\overline{x}, \overline{y})$ is also a source in $\Psi(S_k)$ where $k > n$.*
- *$(\overline{x}, \overline{y})$ where $x < y$ is a true source if $2x < y < \frac{x+n}{2}$*

Theorem (On Connected Components)

Let f_a be a connected component in $\Psi(S_n)$ where $a = \gcd(x, y, n)$ for a given vertex $(\overline{x}, \overline{y}) \in f_a$. Then

- *f_a contains every vertex $(\overline{x}, \overline{y})$ where $\gcd(x, y, n) = a$.*
- *The number of connected components f_a with finite elements in $\Psi(S_n)$ is equal to the number of distinct factors of n.*
- *The number of vertices in f_a is equal to Jordan's totient function $J_k(g)$ evaluated with $k = 2$ and $g = \frac{n}{a}$ where p is prime.*

$$J_2\left(\frac{n}{a}\right) = \left(\frac{n}{a}\right)^2 \prod_{p \mid \frac{n}{a}} \left(1 - \frac{1}{p^2}\right)$$

Theorem (On Cycles)

Let f_a be a connected component in $\Psi(S_n)$. Then

- *f_a contains exactly one cycle of length $\frac{n}{a}$.*
- *A given vertex $(\overline{x}, \overline{y}) \in f_a$ is in the cycle if $x = n - a$ and $\gcd(x, y, n) = a$.*
- *The number of cycles with finite elements in $\Psi(S_n)$ is equal to the number of distinct factors of n.*

References

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