Understanding Energy Dissipation Through Nonlinear Waves on Quantized Vortex Filaments

Kaleigh Rudge, Colorado School of Mines
Advisor: Dr. Scott Strong

Introduction

Nonlinear waves on vortex filaments are fundamental for energy transfer in fluid dynamics and are crucial in the understanding of energy dissipation in quantum turbulence.

Classical Fluid Mechanics

In order to make sense of this energy dissipation in quantum fluids, we must work to better understand how nonlinear waves in quantum fluids relate to classical fluid mechanics, and specifically continuum mechanics. This can be modeled using the Navier-Stokes Equation:

$$\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot (\rho \underline{v} \otimes \underline{v}) = -\nabla p + f \quad (1)$$

Where t is time, p is pressure, ρ is density, $\underline{v}$ is material velocity and f is the gradient of the body potential. This informs how Newtonian fluids move through space. From here, one can relate the Navier-Stokes equation to the Gross-Pitaevskii equation and see relationships between continuum and quantum mechanics.

Derivation for Gross-Pitaevskii Equation

Starting with the Navier-Stokes Equation and expanding the outer product gives:

$$\rho \frac{\partial \underline{v}}{\partial t} + \frac{\partial \rho}{\partial t}\underline{v} + \underline{v}(\nabla \cdot \rho \underline{v}) + \nabla \underline{v} \cdot (\rho \underline{v}) = -\nabla p + f \quad (2)$$

Applying conservation of mass and recognizing that potential flow has the relationship of $\underline{v} = \nabla \phi$, the equation becomes:

$$\rho \frac{\partial \nabla \phi}{\partial t} + \nabla(\nabla \phi) \cdot (\rho \nabla \phi) = -\nabla p + f \quad (3)$$

$$\rho \nabla \frac{\partial \phi}{\partial t} + \rho \nabla \frac{1}{2} ||\nabla \phi||^2 = -\nabla p + f \quad (4)$$

Using the assumption that $\nabla \ln(\rho) \ll \nabla\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\|\nabla \phi\|^2\right)$, then this simplifies to:

$$\nabla\left(\rho \frac{\partial \phi}{\partial t} + \rho \|\nabla \phi\|^2 + p - V\right) = 0 \quad (5)$$

Adding conservation of mass and separating yields:

$$i\left\{\frac{\partial \rho}{\partial t} + \rho \Delta \phi + (\nabla \rho \cdot \nabla \phi)\right\} = \rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho \|\nabla \phi\|^2 + p + V \quad (6)$$

Final Steps

At this point, one multiply through by a constant, γ, say and factor out an $\widetilde{m}$, giving:

$$i\gamma \widetilde{m}\left\{\frac{1}{\widetilde{m}}\frac{\partial \rho}{\partial t} + \frac{\rho}{\widetilde{m}}\Delta \phi + \frac{1}{\widetilde{m}}(\nabla \rho \cdot \nabla \phi)\right\} = \rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho \|\nabla \phi\|^2 + p + V \quad (7)$$

After some manipulation, this gives us the Gross-Pitaevskii Equation:

$$i\hbar \frac{\partial}{\partial t}\psi = \frac{-\hbar^2}{2m}\Delta \psi + g|\psi|^2 \psi + V(x)\psi \quad (8)$$

Where $\gamma \widetilde{m} = \hbar$, which is Planck's constant and $\psi = A\sqrt{\rho}e^{i\eta\theta}$

Discussion and Next Steps

Since the Gross-Pitaevskii Equation governs how particles behave in the quantum continuum, understanding the relationship between the terms is crucial. The addition of γ, gives the energy connection from the classical to quantum continuum. The next steps will be to find connections between phonons in Bose Einstein Condensates and Navier-Stokes. This will help inform the properties of the vortex reconnections and inform the properties of nonlinear waves propagating along the vortex filaments.