A Variant of Hilbert's Nullstellensatz for Graphons

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Overview

- Introduction to graphons
- Introduction to quantum graphs
- Hilbert's Nullstellensatz Applied to Kernels

Introduction to graphons

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- A graphon is a bounded, symmetric, measurable function $W : [0,1]^2 \to [0,1]$.

Introduction to graphons

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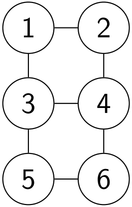

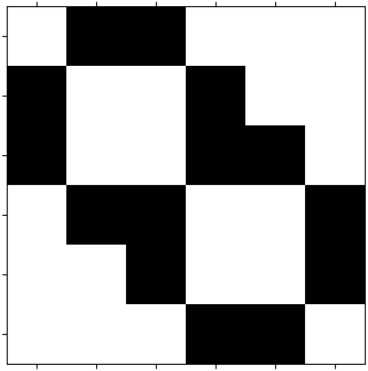

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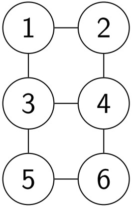

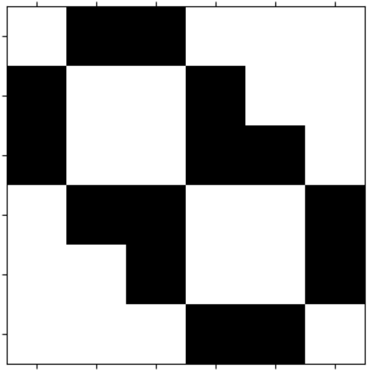

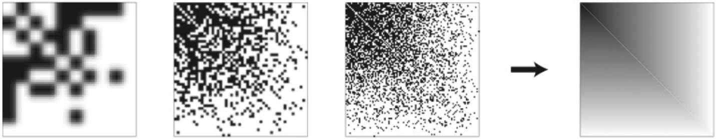

Introduction to quantum graphs

A *quantum graph* is a formal linear combination of multigraphs.
Any quantum graph can be represented as

$$\sum_{i=1}^{n} \alpha_i F_i$$

for real weights α_i and multigraphs F_i.

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We use the "polynomial method," developed by B. Szegedy [2007] to treat graphs and quantum graphs as polynomials. A (quantum) graph g is represented by $\hom(g, X)$.

Introduction to quantum graphs

Homomorphism density is the probability that a randomly chosen map from the vertices of one graph to another preserves edge adjacency.

For a graph G and a graphon W, the homomorphism density is:

$$t(G, W) = \int_{[0,1]^{|V(G)|}} \prod_{ij \in E(G)} W(x_i, x_j) \prod_{i \in V(G)} dx_i.$$

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A set of graphons satisfying a condition of the form $t(g, W) = 0$ for a quantum graph g is called a *graphon variety*.

Hilbert's Nullstellensatz

(**Hilbert's Nullstellensatz**, Hartshorne 2013) Let k be an algebraically closed field, let $\mathfrak{a}$ be an ideal in $A = k[x_1, \ldots, x_n]$, and let $f \in A$ be a polynomial which vanishes at all points of $Z(\mathfrak{a})$. Then $f^r \in \mathfrak{a}$ for some integer $r > 0$.

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Let Q be a set of quantum graphs such that their polynomial representations form an ideal $\mathfrak{a}$.
Let g be a quantum graph such that $\hom(g, X)$ vanishes at all points of $Z(\mathfrak{a})$. Then $\hom(g, X)^r \in \mathfrak{a}$ for some integer $r > 0$ and $t(g, W) = 0$ for all kernels W in the variety defined by Q, $\{W \in \mathcal{W} | t(g, W) = 0 \text{ for all } g \in Q\}$.

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Summary

- Graphons
- Quantum graphs
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Acknowledgements

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Professor Heather Zinn Brooks, Harvey Mudd College

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Harvey Mudd College Mathematics Department

Spare Slides

Present and Future topics of investigation

The Nullstellensatz is a fundamental result in algebraic geometry. What properties change and fall apart as a result of the weaker version?

The Polynomial Representation Method

Let H be a weighted graph on $[q]$ with nodeweights 1 and edgeweights variables x_{ij} (where $x_{ij} = x_{ji}$). We arrange these variables into a symmetric $q \times q$ matrix X. The homomorphism number is represented by a summation over all mappings from $V(G)$ to $[q]$:

$$\hom(G, H) = \sum_{\phi: V(G) \to [q]} \prod_{ij \in E(G)} x_{\phi(i)\phi(j)}$$

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The polynomials g, X, where $g \in \mathcal{Q}$, form the space $\mathbb{C}[X]^{S_q}$ which is the space of polynomials in the polynomial ring $\mathbb{C}[X]$ that are invariant under the permutations of $[q]$.

Some Notation

For every kernel W, the kernel operator $T_W : L_1[0,1] \to L_\infty[0,1]$ is defined by

$$(T_W f)(x) = \int_0^1 W(x,y) f(y)\, dy.$$

If T_W has a finite number of eigenvalues and eigenfunctions, then W is called a *finite rank* kernel.

From here on, we only work with finite ranked kernels.

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Then the set of homomorphism polynomials that correspond to quantum graphs with a zero homomorphism density for a finite rank kernel

$$S = \{\hom(g, X) | t(g, W) = 0, q \in \mathcal{Q}, W \in V\}$$

is an ideal.

Algebraic Kernel Sets

A set Y of finite rank kernels is called an *algebraic set* if:

(a) there exists a subset $T \subseteq \mathbb{C}[X]^{S_q}$ such that $Y = V_q(T)$
or equivalently
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In the space of finite rank kernels, we can define the Zariski Topology by taking the open subsets to be the complements of the algebraic sets.

Some Notation

For any set Y of finite rank kernels, we define the *ideal* of Y in $\mathbb{C}[X]^{S_q}$ by

$$I_q(Y) = \{\hom(g, X) | g \in \mathcal{Q}, t(g, W) = 0 \text{ for all } W \in Y\}.$$

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For a set $Q \subseteq \mathcal{Q}$ of quantum graphs, we define the kernel variety to be

$$V_W(Q) = \{W \in \mathcal{W} | t(g, W) = 0 \text{ for all } g \in Q\}$$

(The set of conditions $t(f_1, W) = 0, \ldots, t(f_n, W) = 0$ is equivalent to the single statement $t(f_1^2 + \ldots + f_n^2, W) = 0$.)

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(d) For any set $Q \subseteq \mathcal{Q}_0$ with a corresponding ideal $\mathfrak{a} = \{\hom(g, X) | g \in Q\} \subseteq \mathbb{C}[X]^{S_q}$, $I_q(V_{\mathcal{W}}(Q)) \supseteq \sqrt{\mathfrak{a}}$, the radical of $\mathfrak{a}$, i.e. $\{\hom(g, X) | t(g, W) = 0 \text{ for all } g \in \mathcal{Q}_0 \text{ and } W \in V_{\mathcal{W}}(Q)\}$ contains the radical of $\mathfrak{a}$. **(This is inclusion or equality**

rather than strict equality)

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The whole space of finite rank kernels is an algebraic set. The node-less graph K_0 has homomorphism polynomial $\hom(K_0, X) = 1$, giving us

$$\mathcal{W} = V_q(1)$$