Monstrous Moonshine: The Monster and the Elliptic Modular J-function

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Introduction

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- Two components:
  - Representation and character theory related to the Monster
  - Klein's J-function
- How do these connect?

Introductory Material

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A sporadic group is a simple group (which means it cannot be "factored") that does not fit nicely into a larger family of simple groups.

The Monster

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$$\begin{aligned} |\mathbb{M}| &= 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \\ &= 808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000 \end{aligned}$$

Representations of Groups

Definition

We define a representation of G over $\mathbb{C}$ as a homomorphism ρ from G to $GL(n, \mathbb{C})$ for some n [3].

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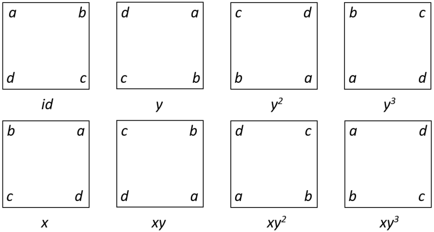
Let us consider D_4, the "square group" (the group of symmetries of a square):



Homomorphism of the Square Group

g	$\rho(g)$	g	$\rho(g)$
id	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	x	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
y	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	xy	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
y^2	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	xy^2	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
y^3	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	xy^3	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Conjugate Representation

"Another" representation:

g	$\sigma(g)$	g	$\sigma(g)$
id	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	x	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
y	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	xy	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
y^2	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	xy^2	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
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$$cl(a) = \{b \in G | b = c^{-1}ac, c \in G\}.$$

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This representation is conjugate to the previous one.

Trace and Character

The trace of a matrix is the sum of its diagonal entries (denoted Tr).

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The character of a group is a function $\chi_{(\rho,V)} : G \to \mathbb{C}$ such that $g \mapsto \mathrm{Tr}(\rho(g))$.
For the square group we were looking at earlier:

The Characters of the Square Group

g	$\rho(g)$	$\sigma(g)$	$\mathrm{Tr}(\rho(g))) = \mathrm{Tr}(\sigma(g)) = \chi(g)$
id	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	2
y	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$	0
y^2	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	-2
y^3	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}$	0

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x	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	0
xy	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	0
xy^2	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	0
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Irreducible Representations

We can break representations down into smaller, irreducible representations.

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Since we are thinking of representations as homomorphisms, this means we can break a representation down into a combination (or direct sum) of irreducible ones.

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The Monster has 194 conjugacy classes, and hence it has 194 equivalence classes of irreducible representations.

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Starting with the trivial representation, the dimensions of the first few irreducible representations are

$$(r_n)_{n=1,\cdots,194} = (1, 196\,883, 21\,296\,876, 842\,609\,326, 18\,538\,750\,076, \cdots).$$

Introduction to the J-Function

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- The definition of the J-function
- The coefficients of the Fourier expansion

Periodic Functions

A function f is elliptic if it is doubly periodic and meromorphic (which means it is complex differentiable on every point of an open subset of the complex plane except for a set of isolated points called poles).

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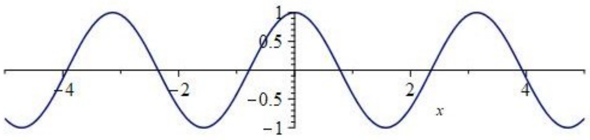
A function is periodic if there is some period ω such that $f(z+\omega) = f(z)$, for z and $z+\omega$ in the domain of f. An example of a periodic function is $\cos 2x$, which has a period of π.



Graph of $y = \cos 2x$

Doubly Periodic Functions

For a function to be doubly periodic it must have two periods, ω_1 and ω_2, with ratio ω_2/ω_1, where this ratio is not real.
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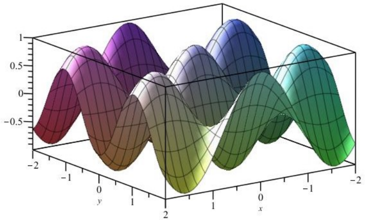
Hence, the periods will go in two different directions.
Here is a function that is not technically doubly periodic, but it does have two periods that go in different directions.



Graph of $f(x, y) = \cos 2x \cdot \cos 3y$

Fundamental Pair of Periods

A pair of periods (ω_1, ω_2) for a function f is a fundamental pair if all the periods of f can be written in the form $m\omega_1 + n\omega_2$ for $m, n \in \mathbb{Z}$.

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A pair of periods (ω_1, ω_2) for a function f is a fundamental pair if all the periods of f can be written in the form $m\omega_1 + n\omega_2$ for $m, n \in \mathbb{Z}$.
We can create a parallelogram by considering $0, \omega_1, \omega_2$, and $\omega_1 + \omega_2$:

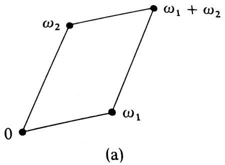
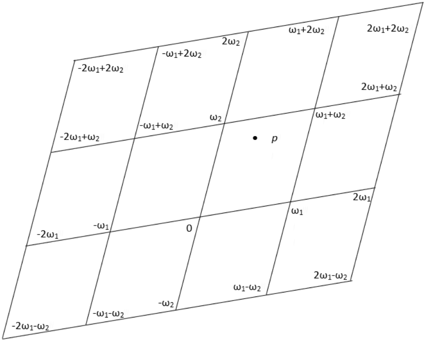


Figure 1.2.a from [1]

Lattice of Periods

This parallelogram is important because it is part of a larger lattice:



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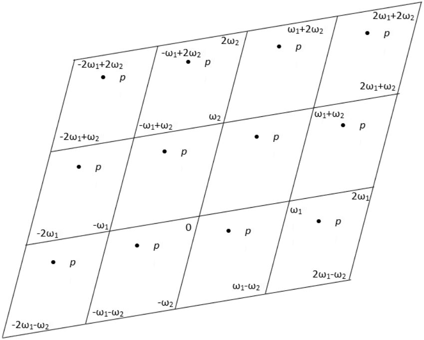

More of the Lattice

Some Notation

- In the discussion that follows, we are going to use the lattice $\Omega = \{\omega = m\omega_1 + n\omega_2 | m, n \in \mathbb{Z}\}$ to define the series $G_n = \sum_{\omega \in \Omega, \omega \neq 0} \frac{1}{\omega^n}$, which is called the Eisenstein series of order n.

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- We define invariants $g_2 = 60G_4$ and $g_3 = 140G_6$.
- The discriminant is a function defined by $\Delta = g_2^3 - 27g_3^2$. There are more natural ways to define Δ (using the Euler function $\phi(q)$ to lead us to $\eta(q)$), but this more artificial construction is helpful for our context.

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- All these functions are homogeneous, which means that for some constant λ,

$$f(\lambda\omega_1, \lambda\omega_2) = \lambda^k f(\omega_1, \omega_2).$$

Klein's J-Function

The J-function is a function of the periods ω_1 and ω_2. We define it as

$$J(\omega_1, \omega_2) = \frac{g_2^3(\omega_1, \omega_2)}{\Delta(\omega_1, \omega_2)} = \frac{\lambda^{-12} g_2^3(\omega_1, \omega_2)}{\lambda^{-12}\Delta(\omega_1, \omega_2)} = \frac{g_2^3(\lambda\omega_1, \lambda\omega_2)}{\Delta(\lambda\omega_1, \lambda\omega_2)} = J(\lambda\omega_1, \lambda\omega_2)$$

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for $\lambda \neq 0$.
For $\tau \in \mathbb{H}$ (where $\tau = \frac{\omega_2}{\omega_1}$ and $\mathbb{H} = \{\tau \in \mathbb{C} | \mathrm{Im}(\tau) > 0\}$),
$J(1, \tau) = J(\omega_1, \omega_2)$ since

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We will denote this function simply as $J(\tau)$ [1].

The Fourier Expansion of $J(\tau)$

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It can be shown that $J(\tau)$ has a Fourier expansion.
For $\tau \in \mathbb{H}$ we have the Fourier expansion

$$12^3 J(\tau) = e^{-2\pi i\tau} + 744 + \sum_{n=1}^{\infty} c(n) e^{2\pi i n\tau},$$

where the $c(n)$ are integers.

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$c(1) = 196\,884$,
$c(2) = 21\,493\,760$, and
$c(3) = 864\,299\,970$.
In some versions of this formula the 744 is subtracted from both sides, and the result is called $J(\tau)$. For $q = e^{2\pi i\tau}$, that gives us

$$12^3 J(\tau) - 744 = q^{-1} + 196\,884q + 21\,493\,760q^2 + 864\,299\,970q^3 + \cdots .$$

From now on we will refer to this version of the formula as $J(\tau)$.

The Monster and the J-Function

Recall that the first few dimensions of irreducible representations of the Monster are

$$(r_n)_{n=1,\cdots,194} = (1, 196\,883, 21\,296\,876, 842\,609\,326, 18\,538\,750\,076, \cdots).$$

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We now notice that

$$\begin{aligned} 196\,884 &= 1 + 196\,883, \\ 21\,493\,760 &= 1 + 196\,883 + 21\,296\,876, \text{ and} \\ 864\,299\,970 &= 2 \cdot 1 + 2 \cdot 196\,883 + 21\,296\,876 + 842\,609\,326, \end{aligned}$$

where the numbers on the left hand side of the equation are coefficients from $J(\tau)$ and the numbers on the right are from the dimensions of the irreducible representations of the Monster.

Conway-Norton's Monstrous Moonshine Conjecture

In 1979 this connection led to the conjecture that there is an infinite-dimensional representation of the Monster with homomorphism $\rho_{\mathbb{M}}$ and vector space $V^\natural = \bigoplus_{i \in \mathbb{Z}, i \geq -1} V_i^\natural$, where if we look at the series

$$T_{[e]} = \sum_{i \geq -1} \mathrm{Tr}(\rho_{\mathbb{M}}(e)|_{V_i^\natural})q^i,$$

(where e is the identity element of $\mathbb{M}$) we get the J function [4].

Brief Historical Overview of Solution

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- Finally, in 1992 Borcherds used Frenkel, Lepowsky, and Meurman's vertex operator algebra and a result from string theory to construct a Lie algebra with infinite dimensions, which he called the monster Lie algebra. He was able to show the proof of Conway and Norton's conjecture could be done using only a finite number of verifications. Hence, Borcherds proved the monstrous moonshine correspondence [4].

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- The J-function has a Fourier expansion with integer coefficients.
- These numbers are closely related.
- The monstrous moonshine conjecture suggested a way this "coincidence" could be explained.
- This conjecture and its proof showed that two different areas of mathematics were surprisingly related.

References

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Thank you to Dr. Darrin Frey for his advice and expertise on this project.

Thank You!

Any Questions?

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For any additional questions feel free to email me at catherineriley@cedarville.edu