Laplacian Simplices Associated to Graphs

Paige Allen and Dr. Marie Meyer

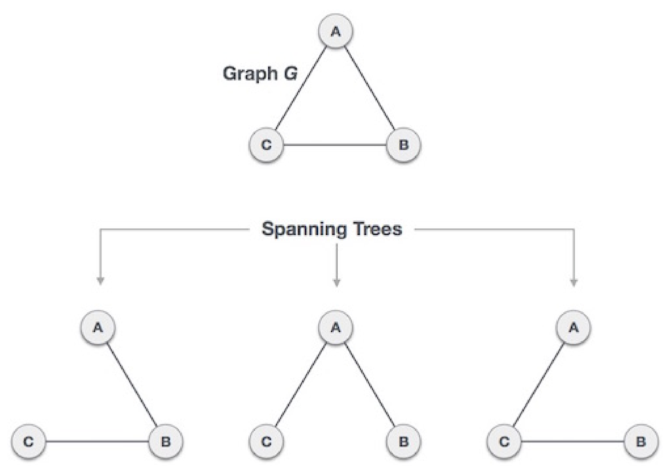
Lewis University

Abstract

This presentation introduces the concept of a Laplacian simplex, which is given from a finite graph by taking the convex hull of the columns of that graph's Laplacian matrix. We define graph terminology along with how to construct the Laplacian matrix. Recently, there is a heightened interest in studying polytopes associated to graphs. It is of interest within the mathematical community to explore how the geometric properties of the polytope relate to the discrete properties of the graph. For instance, the volume of a Laplacian simplex is related to the spanning trees of its underlying graph. We analyze the fundamental parallelepiped of the simplex in order to describe a bijection between lattice points and labeled rooted spanning trees. Finally, we demonstrate the construction of lattice points and present new results regarding the form of lattice points for different families of graphs. This is joint work with Dr. Marie Meyer.

1. Introduction to Graphs

A *graph* G consists of a vertex set $[n] := \{1, \dots, n\}$ and an edge set $E(G) = \{ij \mid i, j \in [n], i \neq j\}$. A *tree* is a graph with no cycles. A *spanning tree* of a graph G is a tree that is a subgraph of G which contains all vertices of G. We let τ denote the number of spanning trees of a graph G. Below is an example of a cycle graph and its spanning trees.



We consider the following $(n \times n)$ matrices associated to a graph G. The *degree matrix* contains the degree of each vertex along its main diagonal and zeros else. The *adjacency matrix* of a graph has entries $a_{ij} = 1$ if $ij \in E(G)$ and 0 else. The *Laplacian matrix* of a graph, denoted $L(G)$, is the difference of the degree and adjacency matrices.

2. Laplacian Simplices

Laplacian simplices are simplices that Braun and Meyer [3] associated to a connected graph G. Let G be a simple connected graph with vertex set $[n]$. The *Laplacian simplex associated to* G, denoted $\mathcal{P}_G$ is the convex hull of the rows of $L(G)$. It is of interest to explore how the properties of the simplex relate to the properties of its underlying graph.

Notable properties of the Laplacian simplex $\mathcal{P}_G$ include

- P_G is $n-1$ dimensional with n vertices (a simplex)
- P_G contains $\vec{0}$ in its strict interior
- Vol $P_G = \sum_{i=1}^{n} \tau = n \cdot \tau$

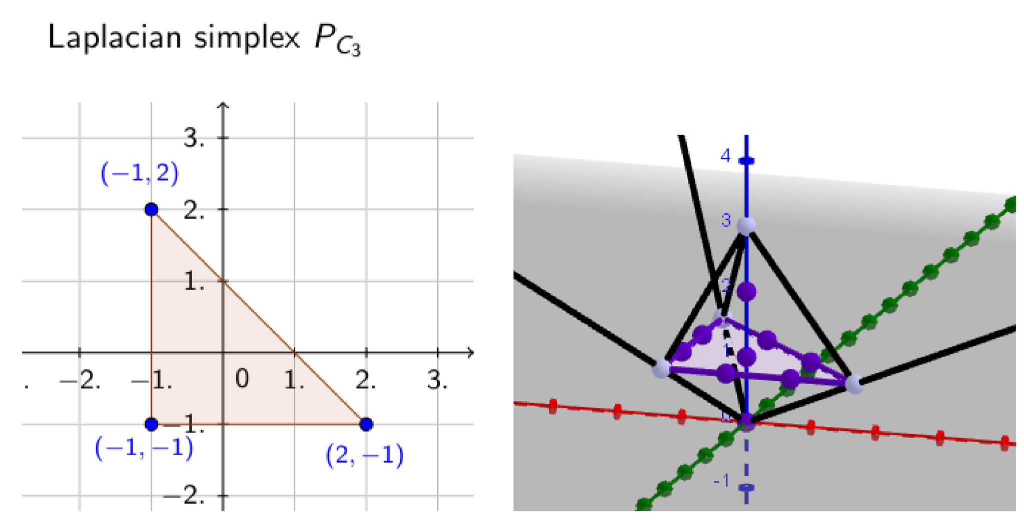
Each simplex is uniquely defined by its *fundamental parallelepiped*, which is defined as

$$\Pi(\mathcal{P}_G) = \{\lambda \cdot [L(n) \mid \mathbb{1}] \mid 0 \leq \lambda_i < 1\}.$$

Notable properties of $\Pi(\mathcal{P}_G)$ include

- $[0, 0, 0, ..., 0, h] \in \Pi_{\mathcal{P}_G}$ for each integer h, $0 \leq h \leq n-1$
- Vol $\mathcal{P}_G = |\Pi_{\mathcal{P}_G} \cap \mathbb{Z}^{n+1}| = n\tau$
- $\vec{\lambda} = \frac{1}{n\tau} \cdot \vec{x}$, where $\vec{x}$ has integer coordinates such that $0 \leq x_i < n\tau$

Pictured below are the Laplacian simplex associated to the 3-cycle graph along with its fundamental parallelepiped.



It is of interest to identify the relationship between properties of the graph and the simplex.

3. New Results

We used the techniques in [1] to discover a way to divide the lattice points of the fundamental parallelepiped into different cosets using addition modulo n. Here is an example of obtaining the different cosets associated with the lattice points of $\Pi(\mathcal{P}_{C_3})$.

$$\Pi(\mathcal{P}_{C_3}) = \{[\lambda_1, \lambda_2, \lambda_3] \cdot \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 1 \end{bmatrix} \Big| 0 \leq \lambda_i < 1, \forall i \in [n]\}$$

Coset 1	Coset 2	Coset 3
$(0,0,0) - (0,0,0)$	$(3,0,6) - (0,-1,1)$	$(0,3,6) - (-1,0,1)$
$(3,3,3) - (0,0,1)$	$(6,3,0) - (1,0,1)$	$(3,6,0) - (0,1,1)$
$(6,6,6) - (0,0,2)$	$(0,6,3) - (-1,1,1)$	$(6,0,3) - (1,-1,1)$

This produces a bijection between the cosets of $\Pi(\mathcal{P}_G)$ and spanning trees of G, as stated in the following result.

Theorem 1 (Allen-Meyer) *The left cosets given by* $\lambda + < \frac{1}{n}\mathbb{1} > \in \Pi(\mathcal{P}_G)$ *are in bijection with the spanning trees of* G.

In [4], the fundamental parallelepiped points of the Laplacian simplex associated to the cycle were classified in the below theorem.

Theorem 2 *For even* n *such that* $n \geq 4$*, the lattice points in* $\Pi(\mathcal{P}_{C_n})$ *are generated from the subgroup*

$$\Lambda(\mathcal{P}_{C_n}) = \frac{1}{2n} < (1, 3, 5, \dots, 2n-1) + 2\mathbb{1} >$$

of the group $< \mathbb{Z}_{2n}^{n}, +_{2n} >$.

We extend these results by considering the wedge graph of a cycle graph and a tree graph to obtain the following two results.

Theorem 3 (Allen-Meyer) *For even natural numbers* $n \geq 4$*, the lattice points in* $\Pi(\mathcal{P}_{C_n \wedge T_n})$ *are of the form* $\vec{\lambda} = \frac{1}{4n} \cdot \vec{x}$*, where* $\vec{x}$ *is in the subgroup*

$$< (1, 5, 9, \dots, 4n-3, (4n-3)\mathbb{1}^{n}) + 2\mathbb{1} >$$

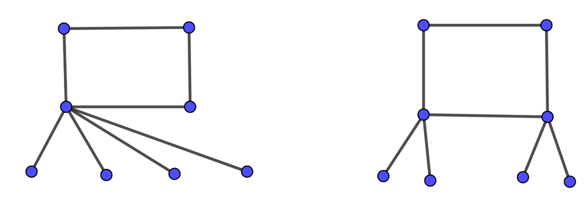
of the group $< \mathbb{Z}_{4n}^{2n}, +_{4n} >$.

Theorem 4 (Allen-Meyer) *For even natural numbers* $n \geq 4$*, the lattice points in* $\Pi(\mathcal{P}_{C_n \wedge T_{\frac{n}{2}} \wedge T_{\frac{n}{2}}})$ *are of the form* $\vec{\lambda} = \frac{\vec{x}}{2n}$*, where* $\vec{x}$ *is in the subgroup*

$$< (1, 3, 5, 7, \dots, 2n-3, 2n-1, (2n-3)\mathbb{1}^{\frac{n}{2}}, (2n-1)\mathbb{1}^{\frac{n}{2}}) + \mathbb{1} >$$

of the group $< \mathbb{Z}_{2n}^{2n}, +_{2n} >$.

Pictured below are the graphs $C_4 \wedge T_4$ and $C_4 \wedge T_2 \wedge T_2$, from left to right, respectively.



4. Future Directions

Here are some questions left to be explored.

Open Problem 5 *Can we define an explicit bijection between the fundamental parallelepiped points and labeled rooted spanning trees of its finite graph?*

Open Problem 6 *Is there a universal way to construct all lambda vectors from a graph's Laplacian matrix so we are able to determine all of the fundamental parallelepiped points within our simplices?*

5. References

[1] V. Batyrev and J. Hofscheier. Lattice polytopes, finite abelian subgroups in (n, C) and coding theory. ArXiv e-prints, Sept. 2013.
[2] M. Beck and S. Robins. Computing the continuous discretely. Undergraduate Texts in Math- ematics. Springer, New York, second edition, 2015.
[3] B. Braun and M. Meyer. Laplacian simplices. Adv. in Appl. Math., 114:101976, 33, 2020.
[4] M. Meyer and T. Pllaha. Laplacian simplices II: A Coding Theoretic Approach. ArXiv e-prints, Sept. 2018.