WASHBURN UNIVERSITY

# An Exploration on a Relation Between arctangents and Fibonacci pairs

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## First Things First: Who is Fibonacci?

Leonardo Pisano, also known as Fibonacci, was one of the greatest mathematicians of the middle ages. Fibonacci was born in 1170 in the city of Pisa. In 1202 he published the book Liber Abaci which introduced Arabic-Hindu numerals (and their properties) to the mathematicians and merchants of Europe. In this book, he introduces a mathematical puzzle known as the “rabbit problem” which is about the breeding patterns of rabbits. The sequence that occurs when the rabbits breed indefinitely, is what is widely known as the Fibonacci sequence. As this research paper pertains to Fibonacci numbers, we felt it was exigent to introduce the mathematician who made this body of work possible (Risley).

## What is the Fibonacci Sequence?

The Fibonacci sequence is as follows: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ... . This is found by adding the first and the second number to get the third (ie. 0+1=1) and then using the second and third number to get the fourth (ie. 1+1=2) and so on, until infinity.

### Recursive Sequence

There is a formula to represent the Fibonacci sequence. It is called the recursive sequence formula for the Fibonacci sequence, and it is as follows:

**Recursive Formula**

$$F_n = F_{n-1} + F_{n-2}$$

### Cassini’s Identity

This is an identity that we needed to use to prove that the upcoming problem related to the Fibonacci sequence.

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

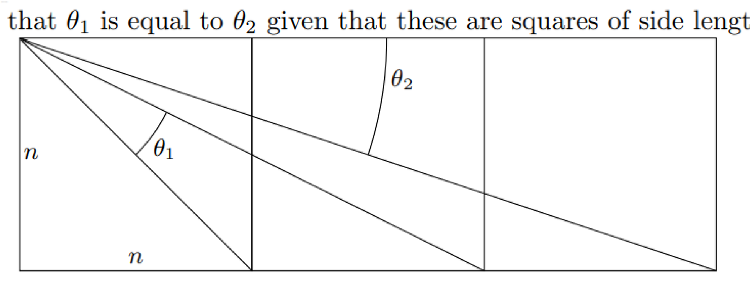
### Binet's Formula

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

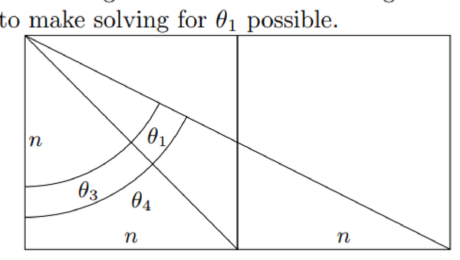
The neat thing about this formula is that we can easily solve for any number in the Fibonacci sequence! (also known as finding the “nth term of the Fibonacci Sequence”)

### THE PROBLEM! (FINALLY)

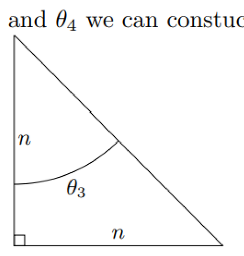
Prove that $\theta_1$ is equal to $\theta_2$ given that these are squares of side length $n$.

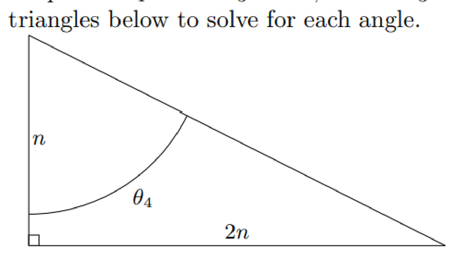



To solve this question we first must find a formula for $\theta_1$ and $\theta_2$. For $\theta_1$ we can first take the above diagram to construct the diagram below. While also adding $\theta_3$ and $\theta_4$ to make solving for $\theta_1$ possible.




We notice that the value of $\theta_1$ must be equal to $\theta_4$ minus $\theta_3$. Now, to find $\theta_3$ and $\theta_4$ we can constuct the two right triangles below to solve for each angle.



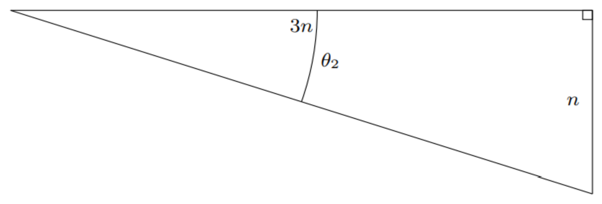




Since these are both right triangles, we can use trigonometry identities to solve for the angle. Using arctangent, we get:

$$\arctan(\frac{2n}{n}) - \arctan(\frac{n}{n}) = \theta_1 \tag{1}$$

$$\arctan(2) - \arctan(1) = \theta_1 \tag{2}$$

Now all we need is an equation for $\theta_2$ and then we can solve. To get $\theta_2$ we can construct a third right triangle.




Using the picture above, we can once again use trigonometry identities to solve for $\theta_2$ using an arctangent.

$$\arctan(\frac{n}{3n}) = \theta_2 \tag{3}$$

$$\arctan(\frac{1}{3}) = \theta_2 \tag{4}$$

Now that we have equations for both $\theta_1$ and $\theta_2$ we can plug them into a calculator to see if they're the same. Sure enough they are.

$$\theta_1 = 63.4349° - 45° = 18.4349° \tag{5}$$

$$\theta_2 = 18.4349° \tag{6}$$

After this discovery, we then asked ourselves if we could generalize this formula to find all possible angles of a chain of square that are equal to eachother. Our first step was to generalize both equations 1 and 3. We did so by first setting them equal to eachother and then replacing 2 with $y$ and 1 with $x$.

Looking back on equation (2):

$$\arctan(2) - \arctan(1) = \theta_1 \tag{2}$$

And equation (4)

$$\arctan(\frac{1}{3}) = \theta_2 \tag{4}$$

We want to find a generalized formula for finding all the possible pairs of equal angles in our square diagram. In order to do this we will substitute y=2 and x=1 in our equations (2) and (4):

$$\arctan(y) + \arctan(x) = \theta_1 \tag{1}$$

$$\arctan(\frac{1}{x+y}) = \theta_2 \tag{2}$$

Next we need to solve for y in terms of x. We will do this to our new version of equation (5):

$$\arctan(y) - \arctan(x) = \arctan(\frac{1}{y+x}) \tag{5}$$

We will take the tangent of both sides and use the the tangent identity

$$\tan(a-b) = \frac{\tan(a) - \tan(b)}{1 - \tan(a)\tan(b)} \tag{3}$$

We end up with the equation

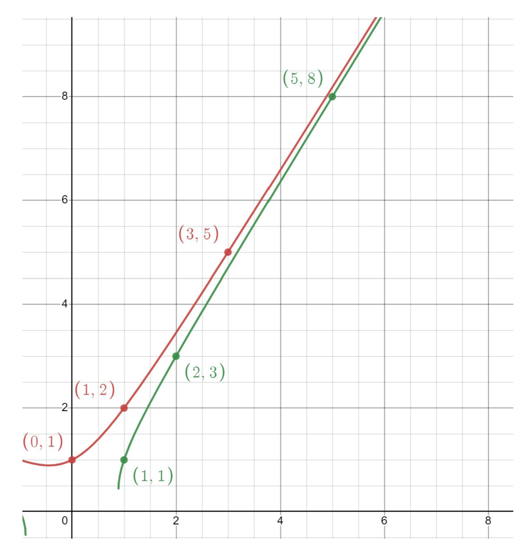
$$y = \frac{x \pm \sqrt{5x^2+4}}{2} \tag{4}$$

Using this equation, if we plug in 1 for x we get 2 for y. If we plug in 3 for x we get 5 for y, BUT it doesn't get between Fibonacci pairs... So, how do we find a formula that can get between Fibonacci pairs. We thought: ”what if the 4 in our equation is considered a constant (called C)?”. What would we get for C if we plugged in 2 for x and 3 for y? C = -4. This new generalized formula

$$y = \frac{x \pm \sqrt{5x^2-4}}{2} \tag{5}$$

can get us from 2 to 3 in the fibonacci sequence!

Now we have two formulas with the ability to generate all fibonacci numbers using only one other fibonacci number. In the past using the Recursive formula, if you wanted to generate all fibonacci numbers, you needed two numbers on the sequence.




### Thank You

To my research partner Gabriel Rose, and to the Washburn University Math Dept. namely, Dr. Porta, Dr, Cook, & Prof. McNamee.

### Works Cited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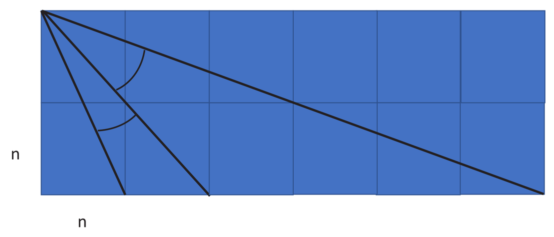
### Future Directions

Would this lead to other Fibonacci-like sequence closed forms?




arctan ( $\frac{2n}{2n}$ ) - arctan ( $\frac{1n}{2n}$ ) = arctan ( $\frac{2n}{6n}$ )

arctan ( y ) - arctan ( x ) = arctan ( $\frac{1}{2(y+x)}$ )

F(x) $\rightarrow$ $y = \frac{x \pm \sqrt{5x^2+4}}{2}$

Even Entries

0 1 1 2 3 5 8 ...

Can be used to get between odd to even Fibonacci numbers

G(x) $\rightarrow$ $y = \frac{x \pm \sqrt{5x^2-4}}{2}$

Odd Entries

0 1 1 2 3 5 8 ...

Can be used to get between even to odd Fibonacci numbers