Preferential and k-Zone Parking Functions

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Abstract

We introduce new variants of parking function rules with backward movement called k-Zone, preferential, and inverse preferential functions. We study the relationship between k-Zone parking functions and k-Naples parking functions and count the number of parking functions under these new parking rules which allow cars that find their preferred spot occupied to back up a certain parameter. One of our main results establishes that the set of non-increasing preference vectors are k-Naples if and only if they are k-Zone. For one of our findings we provide a table of values enumerating these new combinatorial objects in which we discover a unique relationship to the order of the alternating group A_{n+1}, numbers of Hamiltonian cycles on the complete graph, K_n, and the number of necklaces with n distinct beads for $n!$ bead permutations.

Parking Functions

- Imagine n cars enter a one-way street consisting of n parking spots and a list of parking preferences p.
- Each car entering has a preferred spot.
- If that spot is empty, then the car parks.
- If the spot is taken, then there are 5 variances of parking rules that the car can follow.

Consider the following parking preference vector $p = (2, 3, 1, 4)$:

i	p_i	Configuration
1	2	_ c_1 _ _
2	3	_ c_1 c_2 _
3	1	c_3 c_1 c_2 _
4	4	c_3 c_1 c_2 c_4

Motivation

- What are the relationships between different parking functions?
- Are there any connections to other combinatorial objects?

k-Naples Parking Functions

Checks k spots backward one at a time.
Consider the following parking preference vector $p = (4, 4, 3, 2, 4)$:

i	p_i	k	Configuration
1	4	2	_ _ _ c_1 _
2	4	2	_ _ c_2 c_1 _
3	3	2	_ c_3 c_2 c_1 _
4	2	2	c_4 c_3 c_2 c_1 _
5	4	2	c_4 c_3 c_2 c_1 c_5

k-Zone Parking Functions

Checks back immediately k spots for availability.
Consider the following parking preference vector $p = (4, 4, 3, 2, 4)$:

i	p_i	k	Configuration
1	4	2	_ _ _ c_1 _
2	4	2	_ c_2 _ c_1 _
3	3	2	_ c_2 c_3 c_1 _
4	2	2	c_4 c_2 c_3 c_1 _
5	4	2	c_4 c_2 c_3 c_1 c_5

Preferential Parking Functions

Checks $n - i$ spots back for an available spot.
Consider the following parking preference vector $p = (6, 6, 4, 3, 3, 3)$:

i	p_i	$n-i$	Configuration
1	6	5	_ _ _ _ _ c_1
2	6	4	_ _ _ _ c_2 c_1
3	4	3	_ _ _ c_3 c_2 c_1
4	3	2	_ _ c_4 c_3 c_2 c_1
5	3	1	_ c_5 c_4 c_3 c_2 c_1
6	3	0	_ c_5 c_4 c_3 c_2 c_1

Lemma 1. If $n \geq 2$, then $(n, \ldots, n) \in [n]^n$ is not a preferential parking function.

Inverse Preferential Parking Functions

Checks $i - 1$ spots back for an available spot.
Consider the following parking preference vector $p = (6, 6, 4, 3, 3, 3)$:

i	p_i	$i-1$	Configuration
1	6	0	_ _ _ _ _ c_1
2	6	1	_ _ _ _ c_2 c_1
3	4	2	_ _ _ c_3 c_2 c_1
4	3	3	_ _ c_4 c_3 c_2 c_1
5	3	4	_ c_5 c_4 c_3 c_2 c_1
6	3	5	c_6 c_5 c_4 c_3 c_2 c_1

Lemma 2. All preference vectors are inverse preferential parking functions.

k-Zone vs. k-Naples

Consider the following table for $k = 2$. The tables represent the total number of parking functions of k-Naples and of k-Zone for any length n.

For $k = 2$:

n	k-Naples	k-Zone
1	1	1
2	4	4
3	27	27
4	240	244
5	2,731	2,808
6	38,034	39,416
7	627,405	654,302

Conjecture 1. k-Naples is a subset of k-Zone.

Non-Increasing Preference Vectors

Consider the following table of the total number of parking functions formulated from non-increasing preference vector of length n.

For $k = 2$:

n	k-Naples	k-Zone
1	1	1
2	3	3
3	10	10
4	34	34
5	117	117
6	407	407
7	1,430	1,430

Theorem 1. Let $p = (p_1, \ldots, p_n) \in [n]^n$ be a non-increasing preference vector. Then p is a k-Naples if and only if p is a k-Zone.

Enumerating k-Zone

Consider the table enumerating the number of k-Zone parking functions of length n and fixed parameter k.

n	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
1	**1**							
2	3	**4**						
3	16	24	**27**					
4	125	203	244	**256**				
5	1,296	2,225	2,808	3,065	**3,125**			
6	16,807	30,067	39,416	44,424	46,296	**46,656**		
7	262,144	484,071	654,302	757,919	805,543	821,023	**823,543**	
8	4,782,969	9,057,316	12,553,351	14,880,368	16,110,376	16,613,896	16,757,056	**16,777,216**

Coinciding with the OEIS sequence A001710.

Conjecture 2. If $n \geq 2$ and $0 \leq k \leq n-1$, then $|\mathrm{ZPF}(n, k-1)| - |\mathrm{ZPF}(n, k-2)|$ is equal to:

- The order of the alternating group A_{n+1}
- Number of Hamiltonian cycles on the complete graph, K_n
- Number of necklaces with n distinct beads for $n!$ bead permutations.

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