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Partial Sums in Reverse Lexicographic Ordering

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Partitions

- A **Partition** is a way of writing an integer n as a sum of other integers in decreasing order
- **Reverse Lexicographic Ordering** is a way to order partitions from "biggest" to "smallest"
  - *Example:* 7, 61, 52, 511, 43, 421, …

Symmetric Groups

- A **Symmetric Group S_n** is a group of all permutations on n symbols
- The symmetric group S_n is frequently written in **Cycle Notation**, and each has a **Cycle Type**
  - *Example:* (12345) has cycle type 5, (123)(45) has cycle type 32 and (1)(2)(3)(4)(5) has cycle type 11111
- Finally, each permutation has a **Parity**, which can be determined using the **Signum Function**

  $\operatorname{sgn}(\sigma) = (-1)^{n-l(\sigma)}$

  where $l(\sigma)$ represents the length of the partition

Connecting the Two Ideas

- Each cycle type of a permutation of n is a partition of n
- The signum function can assign a positive or negative 1 to each partition
  - $\sigma = (123)(45)$ has cycle type 32 and length 2 → $\operatorname{sgn}(\sigma) = (-1)^{5-2} = (-1)^3 = -1$ → The partition is assigned -1
  - $\sigma = (12345)$ has cycle type 5 and length 1 → $\operatorname{sgn}(\sigma) = (-1)^{5-1} = (-1)^4 = 1$ → The partition is assigned +1

Proposition by Euler

- The sum of the signs of each partition is always nonnegative
  - More specifically, the sum always equals the number of partitions with distinct odd parts

λ	5	41	32	311	221	2111	11111	SUM
Sgn(λ)	1	-1	-1	1	1	-1	1	1

Conjecture by Sheila Sundaram

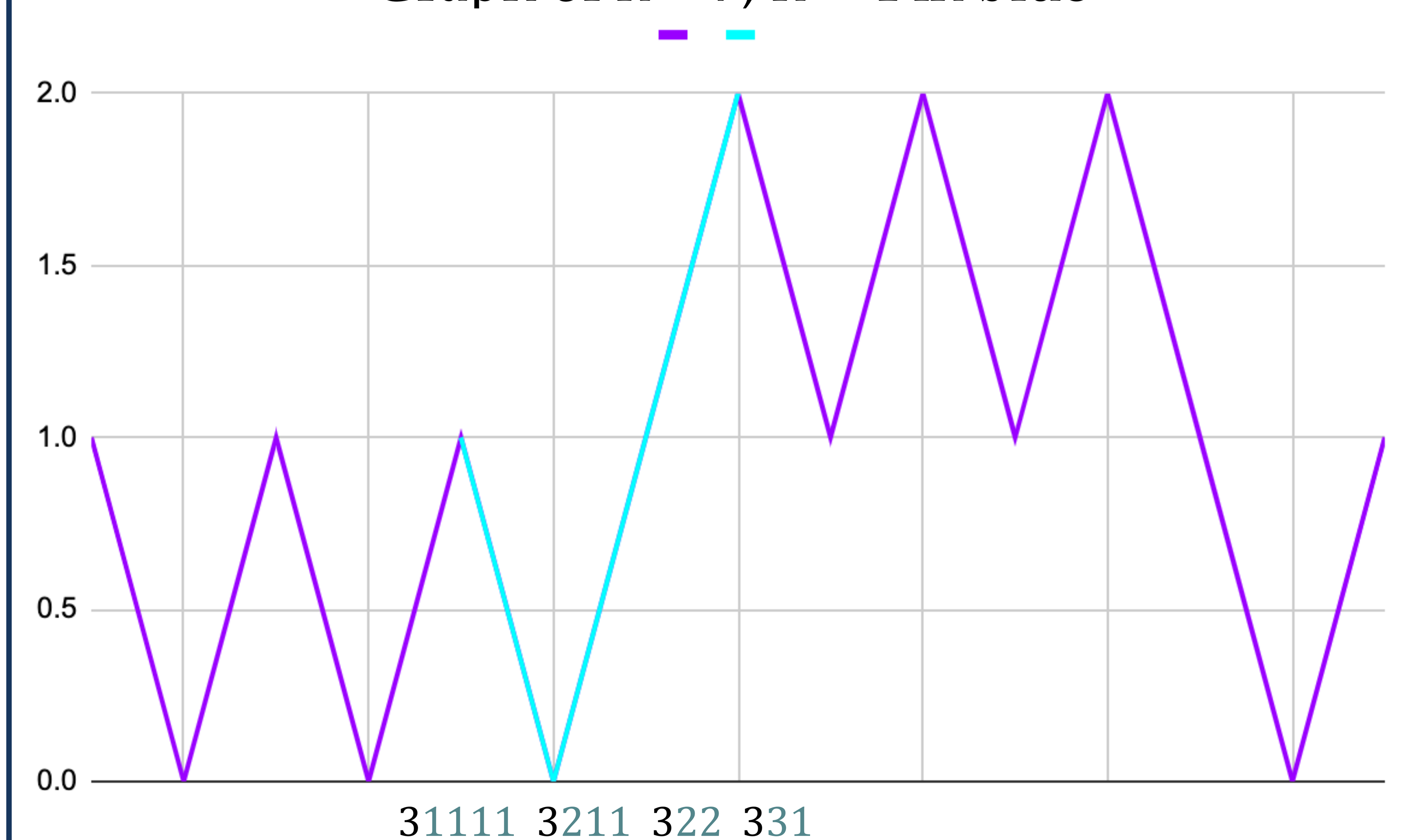
- Using Reverse Lexicographic Ordering and starting at the smallest partition, each partial sum of the partitions is always nonnegative

λ	5	41	32	311	221	2111	11111
Sgn(λ)	1	-1	-1	1	1	-1	1
Partial Sums	1	0	1	2	1	0	1
	←	←	←	←	←	←	←

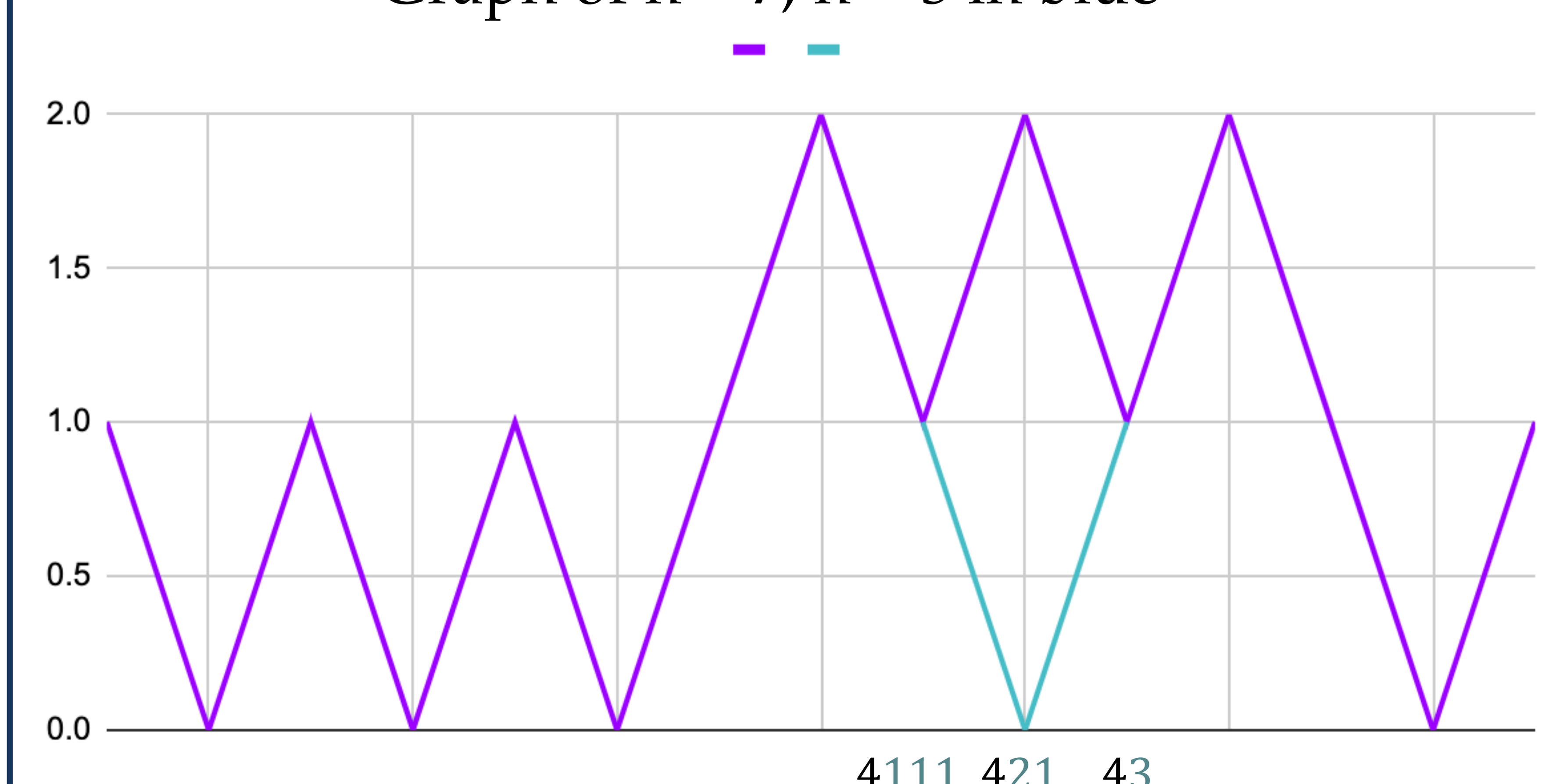
Building Partitions

- Previous partitions can be used to build later partitions
- We can represent this graphically with partitions on x-axis and partial sums on y-axis

Graph of n = 7, n = 4 in blue



Graph of n = 7, n = 3 in blue



- Notice that parts of previous graphs match up perfectly to the later graphs, yet others are flipped to form the new graphs

Result

- When using a previous partition to construct a new partition that starts with an even number, the sign of the new partition is the negation of the sign of the previous partition
  - The smallest partition, the partition made entirely of ones, always has a sign of 1 because $n = l(\lambda)$
  - The next smallest partition replaces two ones with a two, thus making the length one shorter, resulting in a negative sign
  - This pattern continues for all partitions, resulting in the phenomena seen in the graphs

Conclusion

- This conjecture, although still unsolved, will help us uncover the significance of reverse lexicographic ordering and discover possible connections between partial sums and a combinatorial object