THE THREE ROWS GAME AND REPETITIONS IN PAK-STANLEY LABELS

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The G-Shi Arrangement

The *G-Shi arrangement* is given by

$$\mathscr{I}(G) = \{x_i - x_j = 0, 1 \mid \{i, j\} \in E \text{ with } i < j\}.$$

When $G = K_n$, the G-Shi arrangement is the *Shi arrangement*, which has $(n+1)^{n-1}$ regions. Below in the example of the 3-Shi arrangement projected into 2 dimensions:

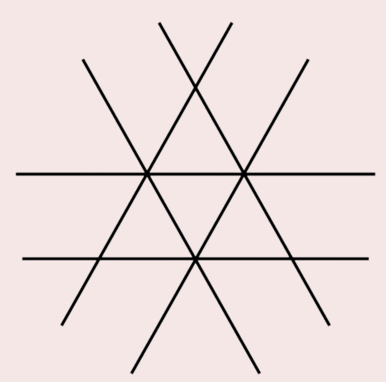


Figure 1: 3-Shi Arrangement

Shi Adjacency Digraph

The G-Shi arrangement endows $\mathbb{R}^n$ with a natural cell structure. The **Shi adjacency digraph** is the 1-skeleton of the dual of that cell structure. The directed edges point away from the center.

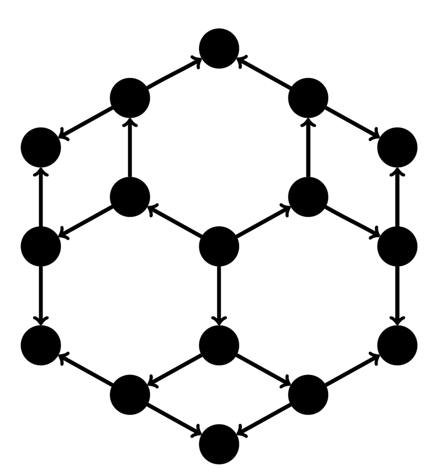


Figure 2: Shi Adjacency Digraph of K_3

Pak-Stanley Labeling

A G-parking function is an n-tuple $(a_0, \ldots, a_{n-1})$ such that for any non-subset $S \subseteq \{0, \ldots, n-1\}$, there exists $v \in S$ such that $a_v < \text{outdeg}_S(v)$.

In the *Pak-Stanley Algorithm*, we label each edge $x_i - x_j = 0$ with i and each edge $x_i - x_j = 1$ with j. Then, starting from R_0, we label the vertices by increasing coordinate i along edges labeled i. This labels each node of the Shi adjacency graph of G with a $G_\bullet$-parking function.

Shi Adjacency Graphs of P_n

A natural place to begin studying repetitions in Pak-Stanley graphs is with path graphs:

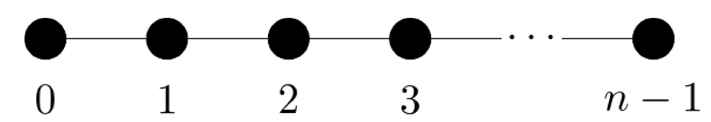


Figure 3: The Path Graph P_n

This is a natural place to start because the Shi adjacency graph of P_n is very simple: it looks like a barycentrically subdivided $(n-1)$-cube.

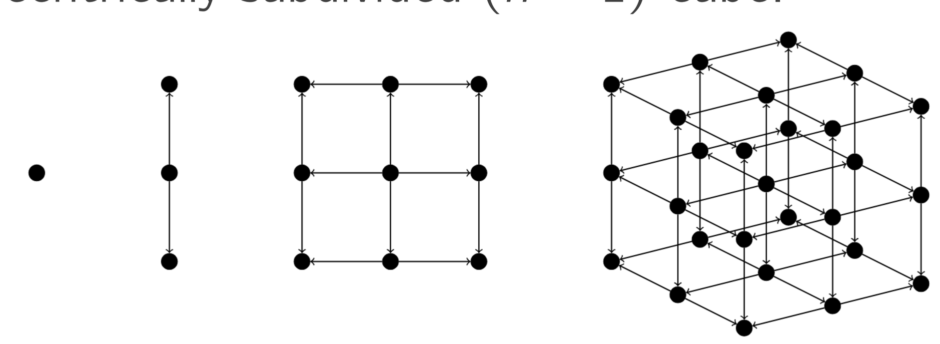


Figure 4: The Shi Adjacency Graphs of P_1, P_2, P_3, and P_4.

The Three Rows Game

The Three Rows Game is played by selecting one square in each column of the following board:

0	1	2	$\cdots$	$n-3$	$n-2$
1	2	3	$\cdots$	$n-2$	$n-1$

Figure 5: The Three Rows Game of P_n

Outcomes (the numbers chosen) and histories (the sequence of moves made) correspond to labels and vertices. This allows us to visualize the behavior of Pak-Stanley labels in 2 rather than $n-1$ dimensions.

Therefore, we can spot arrangements, in particular **patterns**, more easily, allowing us to classify repetitions in the Pak-Stanley Labels using the **Pattern Repetition Theorem**.

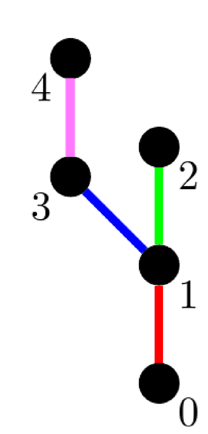
The Pattern Repetition Theorem (C. Bennett, A. Mock, R. Truax)

A *run* of length n is a subsequence of the Pak-Stanley label of the form $0, 1, 1, \ldots, 1, 1, 0$ where there are n 1's. Let $\mathbf{p}$ be a Pak-Stanley label on P_n. If the Pak-Stanley label $\mathbf{p}$ has runs $r_1, \ldots, r_k$ with respective lengths $l_1, \ldots, l_k$, then the number of vertices in $\Gamma_{\mathscr{I}}(P_n)$ with label $\mathbf{p}$ is

$$(l_1 + 1) \cdots (l_k + 1).$$

The T-Three Rows Game

The T-Three Rows Game has a column with an i and a j for each edge $\{i, j\}$.



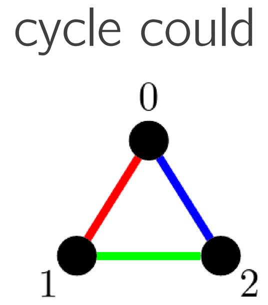
$\Longrightarrow$

0	1	1	3
1	2	3	4

The Importance of Acyclicity

Acyclicity lets us choose any sequence of moves.

$\hookrightarrow$ a cycle could introduce a contradiction.



0	1	0
1	2	2

$x_0 - x_1 < 0$
$x_1 - x_2 < 0$
$x_0 - x_2 > 1$

Since trees have no cycles, this issue doesn't arise.

Results About The Pak-Stanley Labels of All Trees

The Shi adjacency digraphs of all trees have 3^{n-1} vertices. They have 2^{n-1} maximal labels, each of which appear uniquely at "sink" vertices (those of outdegree 0).

Beyond Trees

If G has a cycle, some histories of the G-Three Rows Game are *improper* (they correspond to nonexistent regions). The simplest case is when G *is* a cycle: the cycle graph C_n. In this case, we classify the improper histories to get a "Cycle Repetition Theorem".

Uniqueness of Maximal Labels

A *maximal* $G_\bullet$-parking function is one that cannot be made any larger and remain a $G_\bullet$-parking function. Any maximal $G_\bullet$-parking function appears uniquely.

Future Directions

One of our future goals is to show that finding the number of histories which give a particular outcome **o** given a board $\mathcal{B}$ of the T-Three Rows Game is computationally difficult in a formal sense (perhaps NP-hard). We would also like to show that any maximal $G_\bullet$-parking function appears on a sink of the Shi adjacency graph of G (as this is true for trees).

References

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