# The Harary Index as a Network Reliability Parameter

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## Harary Index

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The **Harary index** of $G$ is the sum of the reciprocals of the distances between every pair of vertices in $V(G)$.

$$H(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{1}{dist(u,v)}$$

where $u \neq v$.

## Path Example

Let $P_n$ be the path graph with $n$ vertices.



The Harary index of a path of length 5 is

$$H(P_5) = 4 \cdot \frac{1}{1} + 3 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} + 1 \cdot \frac{1}{4} = \frac{77}{12}$$

## Definitions

We define $\mathcal{G}(n,m)$, as the set of all graphs with $n$ vertices and $m$ edges.

If $G$ is a graph and $e$ is an edge in $E(G)$, then we denote $G - e$ as the resulting graph when edge $e$ is removed from $E(G)$.

The **change in the Harary index** when an edge $e$ is removed from graph $G$ is denoted as

$$\Delta H(G,e) = H(G) - H(G-e).$$

The **maximum change** in the Harary index with a single edge removal is defined as

$$\Delta^+ H(G) = \max_{e \in E(G)} (\Delta H(G,e)).$$

Similarly, the **minimum change** is defined as

$$\Delta^- H(G) = \min_{e \in E(G)} (\Delta H(G,e)).$$

## Extremal Cases for Edge Removal in $\mathcal{G}(n,m)$

**MIN MIN**

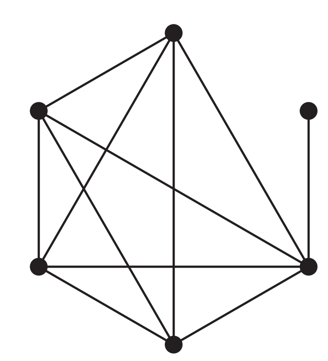
We define the **min-min change** in graph class $\mathcal{G}$ with a single edge removal as

$$\min \Delta^-(\mathcal{G}) = \min_{G \in \mathcal{G}} \Delta^- H(G).$$

This is the minimum change in the Harary index for a single edge removal for all graphs in $\mathcal{G}$.
We found:

$$\min \Delta^-(\mathcal{G}(n,m)) = \begin{cases} 1 & \text{if } m = 1 \text{ and } n \geq 2 \\ \frac{3}{2} & \text{if } m = 2 \text{ and } n = 3 \\ 1 & \text{if } m = 2 \text{ and } n > 3 \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

Ex. $m = 11$, $n = 6$



**MIN MAX**

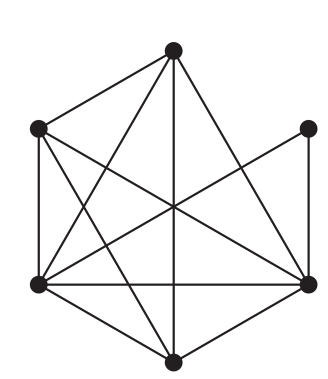
We define the **min-max change** in graph class $\mathcal{G}$ with a single edge removal as

$$\min \Delta^+(\mathcal{G}) = \min_{G \in \mathcal{G}} \Delta^+ H(G).$$

This is the minimum, over all graphs in $\mathcal{G}$, of the maximum change in the Harary index with a single edge removal.
We found:

$$\min \Delta^+(\mathcal{G}(n,m)) = \begin{cases} 1 & \text{if } m = 1 \\ \frac{3}{2} & \text{if } m = 2 \text{ and } n = 3 \\ 1 & \text{if } m = 2 \text{ and } n \geq 4 \\ \frac{2}{3} & \text{if } m = 4 \text{ and } n \geq 4 \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

Ex. $m = 12$, $n = 6$



**MAX MAX**

We define **max-max change** in graph class $\mathcal{G}$ with a single edge removal as

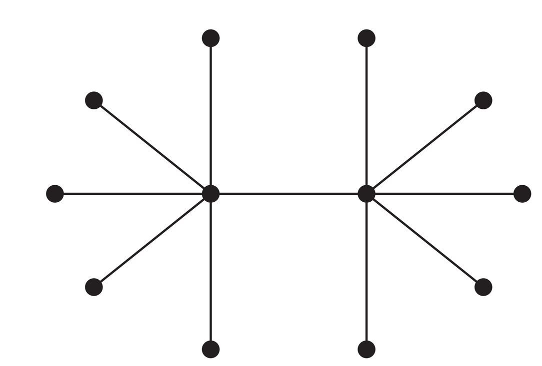
$$\max \Delta^+(\mathcal{G}) = \max_{G \in \mathcal{G}} \Delta^+ H(G).$$

This is the maximum change in Harary index resulting from a single edge removal over all graphs in $\mathcal{G}$.
For any graph $G$ with $n$ vertices, and $a = \lfloor \frac{n-2}{2} \rfloor$, and $b = \lceil \frac{n-2}{2} \rceil$, we found:

$$\Delta^+ H(G) \leq 1 + \frac{n-2}{2} + \frac{ab}{3}.$$

When $m < n$, equality in the above statement holds for the double star graph $D_{a,b}$, so

$$\max \Delta^+(\mathcal{G}(n,m)) = 1 + \frac{n-2}{2} + \frac{ab}{3}.$$



**MAX MIN**

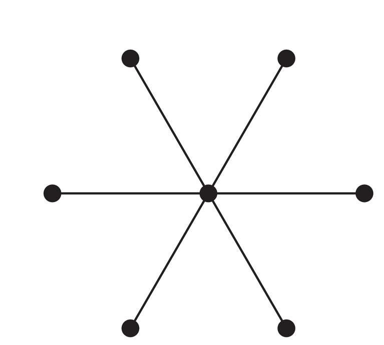
We define the **max-min change** in graph class $\mathcal{G}$ with a single edge removal as

$$\max \Delta^-(\mathcal{G}) = \max_{G \in \mathcal{G}} \Delta^- H(G).$$

This is the maximum, over all graphs in $\mathcal{G}$, of the minimum change in the Harary index with a single edge removal.
With $m < n$, we found:

$$\max \Delta^-(\mathcal{G}(n,m)) = \frac{m+1}{2}.$$

This is satisfied by the star graph, $K_{1,m}$.



## Edge Removals

Let $P_n$ be a **path** graph with $n$ vertices. Then

$$\Delta^- H(P_n) = \sum_{i=1}^{n-1} \frac{1}{i},$$

$$\Delta^+ H(P_n) = \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \left[ \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor + n \bmod 2} \left( \frac{1}{i+j+1} \right) \right].$$

Let $C_n$ be a **cycle** graph with n vertices. Then

$$\Delta^- H(C_n) = \Delta^+ H(C_n) = \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(1 - \frac{i}{n-i}\right).$$

Let $K_{p,q}$ be a **complete bipartite** with $p, q$ vertices. Then

$$\Delta^- H(K_{p,q}) = \Delta^+ H(K_{p,q})$$

$$= \begin{cases} 1 & \text{if } \; p,q = 1 \\ 1 + \frac{q-1}{2} & \text{if } \; p = 1, q > 1 \\ \frac{2}{3} & \text{if } \; p,q \geq 2 \end{cases}$$

Let $K_{a_1,a_2,\ldots,a_\ell}$ be a **complete multipartite** with $n$ vertices and $\ell \geq 3$, and assume $a_i \leq a_j$ for all $i < j$. Then

$$\Delta^- H(K_{a_1,a_2,\ldots,a_\ell}) = \Delta^+ H(K_{a_1,a_2,\ldots,a_\ell}) = \frac{1}{2}.$$

## Future Work

We are working on proving configurations for the max-max case with $m > \binom{a}{2} + \binom{b}{2}$ and the max-min case with $m \geq n$.
We have also started looking at the maximum and minimum change in the Harary index for edge removals in binary trees.
Our findings could also be generalized to the change in the Harary index for vertex removals.

## Acknowledgements

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