Pell and associated Pell braid sequences as GCDs of sums of k consecutive Pell and balancing-related numbers

Janeè Schrader | Faculty Mentor: aBa Mbirika | External Collaborator: Jürgen Spilker (Emeritus, University of Freiburg, Germany)
University of Wisconsin-Eau Claire, Wisconsin

Preliminaries

Definition. The **Pell sequence** $(P_n)_{n\geq 0}$ and the **associated Pell sequence** $(Q_n)_{n\geq 0}$ are defined by the recurrence relations $P_n = 2P_{n-1} + P_{n-2}$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, respectively, with initial conditions $P_0 = 0$, $P_1 = 1$, $Q_0 = 1$, and $Q_1 = 1$.

Remark. The associated Pell (respectively, Pell) sequence is the sequence of numerators (respectively, denominators) of the rational convergents to the square root of 2.

Definition. The **balancing sequence** $(B_n)_{n\geq 0}$ and the **cobalancing sequence** $(b_n)_{n\geq 0}$ are defined by the recurrence relations $B_n = 6B_{n-1} - B_{n-2}$ and $b_n = 6b_{n-1} - b_{n-2} + 2$, respectively, with initial conditions $B_0 = 0$, $B_1 = 1$, $b_0 = 0$, and $b_1 = 0$.

Definition. The **Lucas-balancing sequence** $(C_n)_{n\geq 0}$ and the **Lucas-cobalancing sequence** $(c_n)_{n\geq 0}$ are defined by the recurrence relations $C_n = 6C_{n-1} - C_{n-2}$ and $c_n = 6c_{n-1} - c_{n-2}$, respectively, with initial conditions $C_0 = 1$, $C_1 = 3$, $c_0 = -1$, and $c_1 = 1$.

n	0	1	2	3	4	5	6	7	8	9	10
P_n	0	1	2	5	12	29	70	169	408	985	2378
Q_n	1	1	3	7	17	41	99	239	577	1393	3363
B_n	0	1	6	35	204	1189	6930	40391	235416	1372105	7997214
b_n	0	0	2	14	84	492	2870	16730	97512	568344	3312554
C_n	1	3	17	99	577	3363	19601	114243	665857	3880899	22619537
c_n	−1	1	7	41	239	1393	8119	47321	275807	1607521	9369319

Fig. 1: Table of first 11 terms in sequences $(P_n)_{n\geq 0}$, $(Q_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, $(C_n)_{n\geq 0}$, and $(c_n)_{n\geq 0}$

Braid Sequence

A **braid sequence** arises when we intertwine two sequences. For example in Figure 2, we have the sequence $(P_n)_{n\geq 1}$ in the top row and the associated Pell sequence $(Q_n)_{n\geq 1}$ in the bottom row. The red-dashed path is the sequence $\left(\gcd(B_n, b_n)\right)_{n\geq 1}$ and the blue-solid path is the sequence $\left(\gcd(B_n, b_n+1)\right)_{n\geq 1}$. Both braid sequences are easily proven by the identities

$$B_n = P_nQ_n, \quad b_n = \begin{cases} P_nQ_{n-1} & \text{if } n \text{ is even,} \\ P_{n-1}Q_n & \text{if } n \text{ is odd,}\end{cases} \quad b_n + 1 = \begin{cases} P_{n-1}Q_n & \text{if } n \text{ is even,} \\ P_nQ_{n-1} & \text{if } n \text{ is odd,}\end{cases}$$

and the facts that $\gcd(P_{n-1}, P_n) = 1$ and $\gcd(Q_{n-1}, Q_n) = 1$

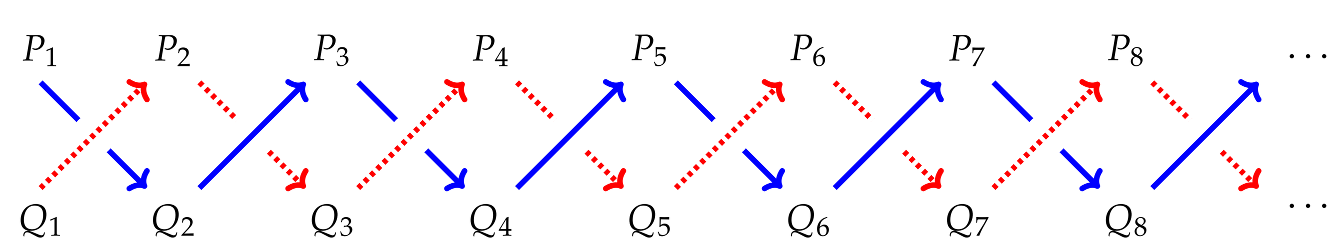


Fig. 2: The braiding of $(P_n)_{n\geq 1}$ and $(Q_n)_{n\geq 1}$

When $(S_n)_{n\geq 0}$ is any of the six sequences $(P_n)_{n\geq 0}$, $(Q_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, $(C_n)_{n\geq 0}$, or $(c_n)_{n\geq 0}$ mentioned earlier, we establish the following notation for $\mathcal{S}^m(k)$, the GCD of all sums of k consecutive m^{th} powers of sequence terms, in the respective six settings:

$$\mathcal{P}^m(k) = \gcd\left\{\left(\sum_{i=1}^{k} P_{n+i}^m\right)_{n\geq 0}\right\} \quad \mathbb{Q}^m(k) = \gcd\left\{\left(\sum_{i=1}^{k} Q_{n+i}^m\right)_{n\geq 0}\right\} \quad \mathcal{B}^m(k) = \gcd\left\{\left(\sum_{i=1}^{k} B_{n+i}^m\right)_{n\geq 0}\right\}$$

$$\mathcal{b}^m(k) = \gcd\left\{\left(\sum_{i=1}^{k} b_{n+i}^m\right)_{n\geq 0}\right\} \quad \mathcal{C}^m(k) = \gcd\left\{\left(\sum_{i=1}^{k} C_{n+i}^m\right)_{n\geq 0}\right\} \quad \mathcal{c}^m(k) = \gcd\left\{\left(\sum_{i=1}^{k} c_{n+i}^m\right)_{n\geq 0}\right\}$$

New Identities

Theorem 1.1. *For $s, r \geq 1$ where s is even, the following identities holds:*

$$P_{s+r} - P_r = \begin{cases} 2P_{\frac{s}{2}}Q_{\frac{s}{2}+r} & \text{if } s \equiv 0 \pmod 4, \\ 2Q_{\frac{s}{2}}P_{\frac{s}{2}+r} & \text{if } s \equiv 2 \pmod 4,\end{cases} \qquad Q_{s+r} - Q_r = \begin{cases} 4P_{\frac{s}{2}}P_{\frac{s}{2}+r} & \text{if } s \equiv 0 \pmod 4, \\ 2Q_{\frac{s}{2}}Q_{\frac{s}{2}+r} & \text{if } s \equiv 2 \pmod 4.\end{cases}$$

Theorem 1.2. *For $k \geq 1$, set $\sigma_S(k,n) := \sum_{i=1}^{k} S_{n+i}$ where $(S_n)_{n\geq 0}$ is any sequence. Then the following identities holds for the five sequences Pell $(P_n)_{n\geq 0}$, associated Pell $(Q_n)_{n\geq 0}$, balancing $(B_n)_{n\geq 0}$, Lucas-balancing $(C_n)_{n\geq 0}$, and Lucas-cobalancing $(c_n)_{n\geq 0}$:*

$$\sigma_P(k,n) = \frac{1}{2}(Q_{n+k+1} - Q_{n+1}) = \begin{cases} 2P_{\frac{k}{2}}P_{\frac{k}{2}+n+1} & \text{if } k \equiv 0 \pmod 4, \\ Q_{\frac{k}{2}}Q_{\frac{k}{2}+n+1} & \text{if } k \equiv 2 \pmod 4,\end{cases} \tag{1}$$

$$\sigma_Q(k,n) = P_{n+k+1} - P_{n+1} = \begin{cases} 2P_{\frac{k}{2}}Q_{\frac{k}{2}+n+1} & \text{if } k \equiv 0 \pmod 4, \\ 2Q_{\frac{k}{2}}P_{\frac{k}{2}+n+1} & \text{if } k \equiv 2 \pmod 4,\end{cases} \tag{2}$$

$$\sigma_B(k,n) = \frac{1}{4}(P_{2k+2n+1} - P_{2n+1}) = \begin{cases} \frac{1}{2}P_kQ_{k+2n+1} & \text{if } k \text{ is even,} \\ \frac{1}{2}Q_kP_{k+2n+1} & \text{if } k \text{ is odd,}\end{cases} \tag{3}$$

$$\sigma_C(k,n) = \frac{1}{2}(Q_{2k+2n+1} - Q_{2n+1}) = \begin{cases} 2P_kP_{k+2n+1} & \text{if } k \text{ is even,} \\ Q_kQ_{k+2n+1} & \text{if } k \text{ is odd,}\end{cases} \tag{4}$$

$$\sigma_c(k,n) = \frac{1}{2}(Q_{2k+2n} - Q_{2n}) = \begin{cases} 2P_kP_{k+2n} & \text{if } k \text{ is even,} \\ Q_kQ_{k+2n} & \text{if } k \text{ is odd.}\end{cases} \tag{5}$$

Theorem 1.3. *For $k \geq 1$, set $\sigma_b(k,n) := \sum_{i=1}^{k} b_{n+i}$. Then the following identity holds:*

$$\sigma_b(k,n) = \frac{1}{2}(B_{k+n} - B_n - k). \tag{6}$$

Observation

Consider $\left(\sum_{i=1}^{k} P_{n+i}\right)_{n\geq 0}$ when $k = 6$. Then we have

$$\begin{aligned} P_1 + P_2 + P_3 + P_4 + P_5 + P_6 &= 119 \\ P_2 + P_3 + P_4 + P_5 + P_6 + P_7 &= 287 \\ P_3 + P_4 + P_5 + P_6 + P_7 + P_8 &= 693 \\ P_4 + P_5 + P_6 + P_7 + P_8 + P_9 &= 1673 \\ P_5 + P_6 + P_7 + P_8 + P_9 + P_{10} &= 4039 \\ &\vdots \end{aligned}$$

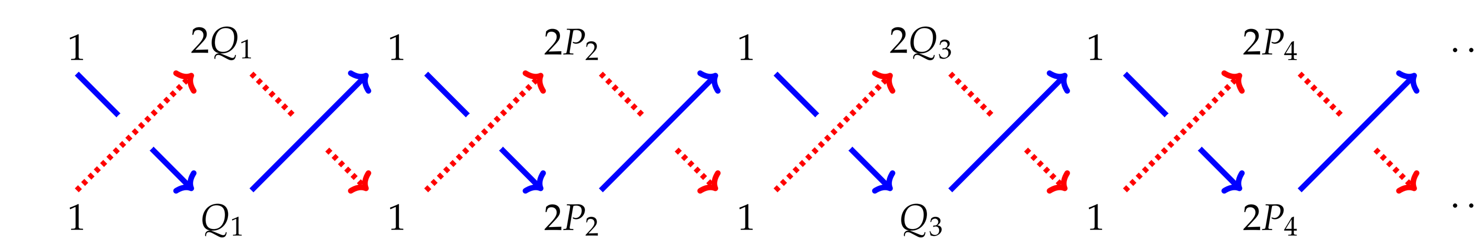
Look!
$7 \cdot 17 = 119$
$7 \cdot 41 = 287$
$7 \cdot 99 = 693$
$7 \cdot 239 = 1673$
$7 \cdot 577 = 4039$

This pattern goes on for infinitely many sums with all of them being divisible by 7. Furthermore, 7 is the greatest integer that divides all sums. This value is $\mathcal{P}^1(6)$. Moreover, $\mathcal{P}^1(6)$ equals Q_3.

Further observe that the sequence $17, 41, 99, 239, 577, \ldots$ is equal to $Q_4, Q_5, Q_6, Q_7, Q_8, \ldots$. This holds by Identity (1) of Theorem 1.2.

Results for $\left(\mathcal{P}^1(k)\right)_{k\geq 1}$ and $\left(\mathbb{Q}^1(k)\right)_{k\geq 1}$

These are the braids for braid sequences $\left(\mathcal{P}^1(k)\right)_{k\geq 1}$ in blue and $\left(\mathbb{Q}^1(k)\right)_{k\geq 1}$ in dashed red.

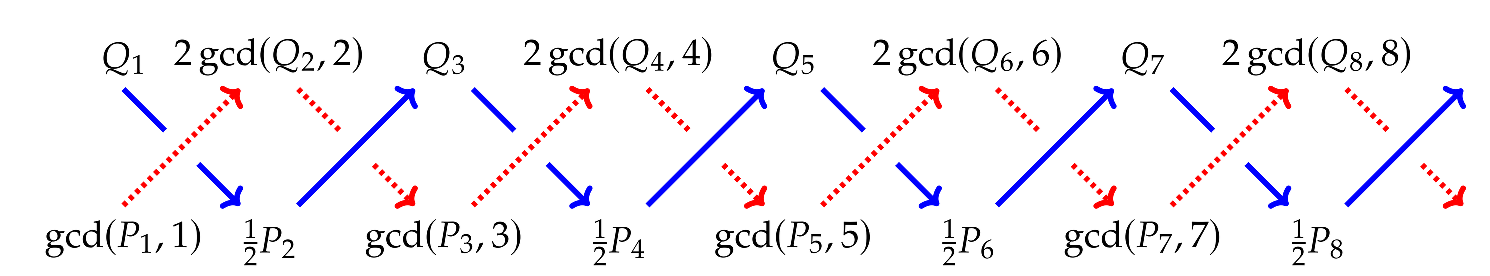


Theorem 1.4. *For all $k \geq 1$, we have*

$$\mathcal{P}^1(k) = \begin{cases} 2P_{\frac{k}{2}} & \text{if } k \equiv 0 \pmod 4, \\ Q_{\frac{k}{2}} & \text{if } k \equiv 2 \pmod 4, \\ 1 & \text{if } k \equiv 1,3 \pmod 4,\end{cases} \qquad \mathbb{Q}^1(k) = \begin{cases} 2P_{\frac{k}{2}} & \text{if } k \equiv 0 \pmod 4, \\ 2Q_{\frac{k}{2}} & \text{if } k \equiv 2 \pmod 4, \\ 1 & \text{if } k \equiv 1,3 \pmod 4.\end{cases}$$

Results for $\left(\mathcal{B}^1(k)\right)_{k\geq 1}$ and $\left(\mathcal{b}^1(k)\right)_{k\geq 1}$

These are the braids for braid sequences $\left(\mathcal{B}^1(k)\right)_{k\geq 1}$ in blue and $\left(\mathcal{b}^1(k)\right)_{k\geq 1}$ in dashed red.

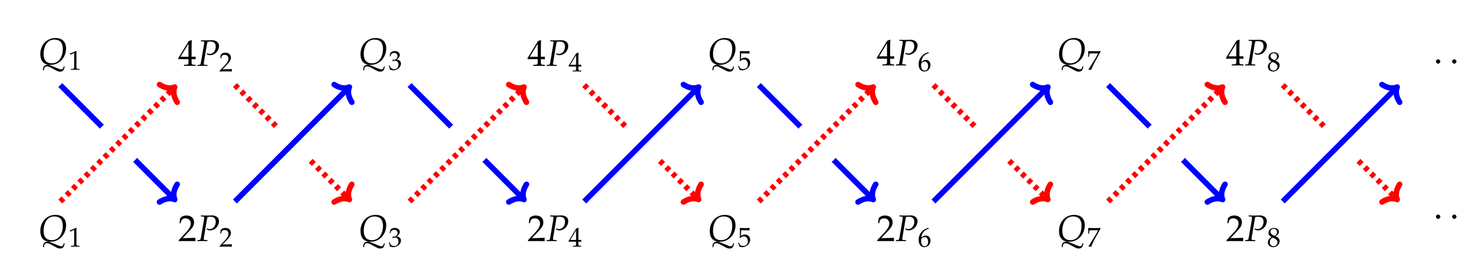


Theorem 1.5. *For all $k \geq 1$, we have*

$$\mathcal{B}^1(k) = \begin{cases} \frac{1}{2}P_k & \text{if } k \text{ is even,} \\ Q_k & \text{if } k \text{ is odd,}\end{cases} \qquad \mathcal{b}^1(k) = \begin{cases} \gcd(P_k, k) & \text{if } k \text{ is even,} \\ 2\gcd(Q_k, k) & \text{if } k \text{ is odd.}\end{cases}$$

Results for $\left(\mathcal{C}^1(k)\right)_{k\geq 1}$ and $\left(\mathcal{c}^1(k)\right)_{k\geq 1}$

These are the braids for braid sequences $\left(\mathcal{C}^1(k)\right)_{k\geq 1}$ in blue and $\left(\mathcal{c}^1(k)\right)_{k\geq 1}$ in dashed red.



Theorem 1.6. *For all $k \geq 1$, we have*

$$\mathcal{C}^1(k) = \begin{cases} 2P_k & \text{if } k \text{ is even,} \\ Q_k & \text{if } k \text{ is odd,}\end{cases} \qquad \mathcal{c}^1(k) = \begin{cases} 4P_k & \text{if } k \text{ is even,} \\ Q_k & \text{if } k \text{ is odd.}\end{cases}$$

Future Research

1. Complete $\mathcal{S}^2(k)$ for all 6 sequences.
2. Determine $\mathcal{S}^m(k)$ where $m > 2$ for all 6 sequences.
3. Find the closed forms for the values $\gcd(S_k, k)$ for all 6 sequences.

References

[1] A. Behera and G. K. Panda, On the square roots of triangular numbers, *Fibonacci Quart.* **37** (1999), no. 2, 98–105.

[2] D. Guyer and a. Mbirika, GCD of sums of k consecutive Fibonacci, Lucas, and generalized Fibonacci numbers, *J. Integer Seq.* **24** (2021), no. 9, Article 21.9.8, 25 pp.

[3] T. Koshy, *Pell and Pell-Lucas Numbers with Applications*, Springer, 2014.

[4] a. Mbirika and J. Spilker, GCD of sums of k consecutive squares of generalized Fibonacci numbers, *Fibonacci Quart.* **60** (2022), no. 5, 255–269.

Acknowledgements: We would like to thank the UW-Eau Claire mathematics department and the UW-Eau Claire Office of Research and Sponsored Programs.