Subseries of the Harmonic Series

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Abstract: It is well known that the harmonic series $\sum \frac{1}{n}$ diverges. We consider two subseries of $\sum \frac{1}{n}$. The first subseries consists of the sum of all fractions whose denominator n contains no digit 9. The surprising result is that the remaining subseries converges. For the second series, we consider the sum of all fractions whose denominators are pandigital numbers. A pandigital number is a positive integer that contains at least one of each of the ten digits. In this case, we show that this subseries is divergent. My research project extends some of these results to other subseries of divergent series.

The Harmonic Series is given by

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

The series diverges.

Harmonic series where no denominator has digit 9

$$\sum_{no\ digit\ 9\ in\ n} \frac{1}{n}$$

$$\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{8}+\frac{1}{10}+\cdots+\frac{1}{18}+\frac{1}{20}+\cdots+\frac{1}{28}+\cdots+\frac{1}{80}+\cdots\frac{1}{88}+\frac{1}{100}+\cdots\frac{1}{108}+\frac{1}{110}+\frac{1}{118}+\cdots$$

Question: Does this series converge or diverge?

Let a_i = sum of the unit fractions where each denominator contains i digits, none with a 9 in it

Start with i=1:

$$a_1 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{8} < 1 + 1 + 1 + \cdots + 1 = 8$$

$$a_2 = \left(\frac{1}{10} + \cdots + \frac{1}{18}\right) + \left(\frac{1}{20} + \cdots + \frac{1}{28}\right) + \cdots + \left(\frac{1}{80} + \cdots + \frac{1}{88}\right)$$

$$< \underbrace{\left(\frac{1}{10} + \cdots + \frac{1}{10}\right)}_{\text{9 fractions}} + \underbrace{\left(\frac{1}{10} + \cdots + \frac{1}{10}\right)}_{\text{9 fractions}} + \cdots + \underbrace{\left(\frac{1}{10} + \cdots + \frac{1}{10}\right)}_{\text{9 fractions}}$$

$$= \underbrace{9\left(\frac{1}{10}\right) + 9\left(\frac{1}{10}\right) + \cdots + 9\left(\frac{1}{10}\right)}_{\text{8 products}}$$

$$= 8\left(\frac{9}{10}\right)$$

Using the same idea for a_3, there are 8×9^2 reciprocals, each less than or equal to $\frac{1}{100}$, therefore

$$a_3 < 8 \times 9^2 \left(\frac{1}{100}\right) = 8\left(\frac{9}{10}\right)^2$$

In general,

$$a_i < 8\left(\frac{9}{10}\right)^{i-1}$$

$$\sum_{\text{no digit 9 in } n} \frac{1}{n} = a_1 + a_2 + a_3 + \cdots$$

$$< 8(1) + 8\left(\frac{9}{10}\right) + 8\left(\frac{9}{10}\right)^2 + \cdots$$

geometric series

$$= \frac{8}{1-\left(\frac{9}{10}\right)} = \frac{8}{\left(\frac{1}{10}\right)} = 80$$

first term: 8

ratio: $\frac{9}{10} < 1$

Geometric series converges implies

$$\sum_{\text{no digit 9 in } n} \frac{1}{n}$$

converges by the comparison test.

A closer look at

$$\sum_{no\ digit\ 9\ in\ n} \frac{1}{n}$$

- Let K = some positive integer
- N_K= positive integers less than or equal to K with no digit equal to 9
- M_K= positive integers less than or equal to K with at least one digit equal to 9
- $K = |N_K| + |M_K|$

- Example: $K = 35$

- $N_{35} = \{1, 2, \dots, 8, 10, 11, \dots, 18, 20, 21, \dots, 28, 30, 31, \dots, 35\}$ $\quad |N_{35}| = 32$
- $M_{35} = \{9, 19, 29\}$ $\quad |M_{35}| = 3$
- $K = 35 = 32 + 3 = |N_{35}| + |M_{35}|$

Question: For larger and larger values of K, how do $|M_K|$ and $|N_K|$ compare to each other?

We shall look at $\lim_{K\to\infty} \frac{|N_K|}{|M_K|}$.

We found that if $K = 10^k$, then $\left|M_{10^k}\right| = 10^k - 9^k$

Assume $K = 10^k$ and $\left|M_{10^k}\right| = 10^k - 9^k$. Since $K = |N_K| + |M_K|$,

$$\lim_{K\to\infty}\frac{|N_K|}{|M_K|} = \lim_{K\to\infty}\frac{K - |M_K|}{|M_K|} = \lim_{k\to\infty}\frac{10^k - \left(10^k - 9^k\right)}{\left(10^k - 9^k\right)} = \lim_{k\to\infty}\frac{9^k}{\left(10^k - 9^k\right)}\cdot\frac{\frac{1}{9^k}}{\frac{1}{9^k}}$$

$$= \lim_{k\to\infty}\frac{1}{\frac{10^k}{9^k} - 1} = \lim_{k\to\infty}\frac{1}{\left(\frac{10}{9}\right)^k - 1} = 0$$

As K gets larger and larger, there are more positive integers less than or equal to K with 9 as one of its digits than those that have no digit 9.

Definition: A positive integer is *pandigital* if it contains each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 at least once and whose leading digit is nonzero.

Interesting Properties:

1. The smallest pandigital number is 1023456789

2. A ten-digit pandigital number is always divisible by 9 since the sum of the digits is always 45

3. An interesting relation exists between these pandigital numbers:

$$\frac{9876453120 \cdot 9876543210}{1234567890 \cdot 7901234568} = 10$$

4. The smallest pandigital prime number in base 10 is 10123457689

Consider the following series:

$$\sum_{p\ (pandigital)} \frac{1}{p}$$

Question: Does this series converge or diverge?

Let $a = 1234567890$ and $n = 10^{10}r + a$ where $r \in \mathbb{N} \cup \{0\}$.

$$a \quad = 1\,2\,3\,4\,5\,6\,7\,8\,9\,0$$

$$10^{10} = 1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0$$

$n =$	$10^{10}r$	a
	leftmost digits	rightmost digits

We note that $n = 10^{10}r + a$ does not include all pandigital numbers.

Example: If $b = 2134567890$, then $10^{10}r + b$ is not of the form given by n.

Furthermore, $a < 10^{10}$ and therefore $\frac{a}{10^{10}} \leq 1$. Assume $n = 10^{10}r + a$.

$$\sum_{p\ pandigital} \frac{1}{p} \geq \sum_{r=0}^{\infty} \frac{1}{n} = \sum_{r=0}^{\infty} \frac{1}{10^{10}r + a} = \frac{1}{10^{10}} \sum_{r=0}^{\infty} \frac{1}{r + \frac{a}{10^{10}}} \geq \frac{1}{10^{10}} \sum_{r=0}^{\infty} \frac{1}{r+1}$$

↑

Diverges

(Harmonic Series)

Using the comparison test,

$$\sum_{p\ pandigital} \frac{1}{p} \text{ diverges}.$$

Research Questions

1. Consider p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $0 < p \leq 1$

Special case: $p = 1$ (Harmonic series)

Conjecture: $\sum_{no\ digit\ 9\ in\ n}^{\infty} \frac{1}{n^p}$ converges <=> $log_{10}(9) < p \leq 1$

2. Consider series: $\sum^{\infty} \frac{n}{an+1}$ where $0 < a$

Series diverges since $\lim_{n \to \infty} \frac{n}{an+1} = \frac{1}{a} \neq 0$

Any a-values for which $\sum_{no\ digit\ 9}^{\infty} \frac{n}{an+1}$ converges?

Any questions?

References

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