N-Potents in Commutative Rings

Delaney Morgan

Wellesley College

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1 Some ring theory
2 Introduction to the Project
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**Defn:** A ring, $R$, is a nonempty set together with two binary operations, addition and multiplication, such that for all $a, b, c \in R$:

- Commutativity of addition: $a + b = b + a$
- Associativity of addition: $(a + b) + c = a + (b + c)$
- Additive identity: There is an additive identity, denoted 0, such that $a + 0 = a$
- Additive inverse: For all $a$, there exists an additive inverse $-a$ such that $a + (-a) = 0$
- Associativity of multiplication: $(ab)c = a(bc)$
- Distributive property: $a(b + c) = ab + ac$

**Defn:** $R$ is a commutative ring if $ab = ba$ for all $a, b \in R$.

**Defn:** A ring $R$ is unital if there exists some unity element 1 such that $a \cdot 1 = a$ for all $a \in R$. 

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- $a + b = b + a$ (commutativity of addition)
- $(a + b) + c = a + (b + c)$ (associativity of addition)
- There is an additive identity, denoted 0, such that $a + 0 = a$
- For all $a$ there exists an additive inverse $-a$ such that $a + (-a) = 0$
- $a(bc) = (ab)c$ (associativity of multiplication)
- $a(b + c) = ab + bc$ (distributive property)
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Defn: $R$ is a *commutative ring* if $ab = ba$ for all $a, b \in R$

Defn: a ring $R$ is *unital* if there exists some unity element 1 such that $a \cdot 1 = a$ for all $a \in R$
Defn: a subring $I$ of a ring $R$ is called an *ideal* of $R$ if for every $r \in R$ and every $a \in I$, $ra$ and $ar$ are in $I$.
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**Example:** $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z} = \{0 + 4\mathbb{Z}, 1 + 4\mathbb{Z}, 2 + 4\mathbb{Z}, 3 + 4\mathbb{Z}\}$
**Defn:** Let $R_1, R_2$ be rings. Then the *direct product* of $R_1$ and $R_2$ is the Cartesian product

$$R_1 \times R_2 = \{(r_1, r_2) \mid r_1 \in R_1, r_2 \in R_2\}$$

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Defn: a \textit{subdirect product} is a subring of a direct product
**Defn**: We say $e \in R$ is an *n-potent* if $e^n = e$ for some positive integer $n$. When $n = 2$ we call $e$ *idempotent* and when $n = 3$ we call $e$ *tripotent*. 

**Example**: Consider $\mathbb{Z}_4$. 0 and 1 are idempotent. 0, 1, and 3 are tripotent. 0 and 2 are nilpotent.
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Questions?
A few remarks

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Throughout this presentation, when we say an integer $n$ is in $R$, what we really mean is that $n \cdot 1$ is in $R$
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Define $R$ to be a unital, commutative ring such that for every $a \in R$, there exists $x_1, x_2, b \in R$ with $x_1^2 = x_1$, $x_2^3 = x_2$, and $b^m = 0$ for some $m$, such that $a = x_1 + x_2 + b$. The goal of the project is to develop a description of $R$. 
Research Goals

The goal of my project is to classify rings where every element can be written as the sum of one idempotent, one tripotent, and one nilpotent.

Define \( R \) to be a unital, commutative ring such that for every \( a \in R \), there exists \( x_1, x_2, b \in R \) with \( x_1^2 = x_1 \), \( x_2^3 = x_2 \), and \( b^m = 0 \) for some \( m \), such that \( a = x_1 + x_2 + b \). The goal of the project is to develop a description of \( R \).

Research has been done studying sums of idempotents, sums of tripotents, and sums of higher-powered \( n \)-potents, but relatively little work has been done on sums of mixed-powered potents.
We found that $6 \in R$ is a nilpotent element. So $6^k = 0$ for some $k$.

$6 = 2 \cdot 3$, so this implies that $R \cong R/2^k R \times R/3^k R$

Note 2 is nilpotent in $R/2^k R$ and 3 is nilpotent in $R/3^k R$
We have $R \cong R/2^k R \times R/3^k R$, we can now observe these rings more directly. Take $R_2 = R/2^k R$ and $R_3 = R/3^k R$. 

Let $N(R)$ denote the ideal of all nilpotent elements in $R$. We find that $R_2 / N(R)$ is a subdirect product of copies of $\mathbb{Z}_2$ and $R_3 / N(R)$ is a subdirect product of copies of $\mathbb{Z}_3$. So we conclude that $R / N(R)$ is a subdirect product of copies of $\mathbb{Z}_2$ and $\mathbb{Z}_3$. 

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So we conclude that $R/N(R)$ is a subdirect product of copies of $\mathbb{Z}_2$ and $\mathbb{Z}_3$
Thank you to the NCUWM organizers, and to my advisor, Alex Diesl

Thank you for attending!

Questions?