N-Potents in Commutative Rings

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January 21, 2023

Outline

1. Some ring theory
2. Introduction to the Project
3. Results

Rings

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- $a + b = b + a$ (commutativity of addition)
- $(a + b) + c = a + (b + c)$ (associativity of addition)
- There is an additive identity, denoted 0, such that $a + 0 = a$
- For all a there exists an additive inverse $-a$ such that $a + (-a) = 0$
- $a(bc) = (ab)c$ (associativity of multiplication)
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Defn: R is a *commutative* ring if $ab = ba$ for all $a, b \in R$

Defn: a ring R is *unital* if there exists some unity element 1 such that $a \cdot 1 = a$ for all $a \in R$

Quotient Rings

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Example: $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z} = \{0 + 4\mathbb{Z}, 1 + 4\mathbb{Z}, 2 + 4\mathbb{Z}, 3 + 4\mathbb{Z}\}$

Products of Rings

Defn: Let R_1, R_2 be rings. Then the *direct product* of R_1 and R_2 is the Cartesian product

$$R_1 \times R_2 = \{(r_1, r_2) \mid r_1 \in R_1, r_2 \in R_2\}$$

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Defn: a *subdirect product* is a subring of a direct product

Potent Elements

Defn: We say $e \in R$ is an *n-potent* if $e^n = e$ for some positive integer n. When $n = 2$ we call e *idempotent* and when $n = 3$ we call e *tripotent*.

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Questions?

A few remarks

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Throughout this presentation, when we say an integer n is in R, what we really mean is that $n \cdot 1$ is in R

Research Goals

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Research has been done studying sums of idempotents, sums of tripotents, and sums of higher-powered n-potents, but relatively little work has been done on sums of mixed-powered potents

Results

We found that $6 \in R$ is a nilpotent element. So $6^k = 0$ for some k.

$6 = 2 \cdot 3$, so this implies that $R \cong R/2^k R \times R/3^k R$

Note 2 is nilpotent in $R/2^k R$ and 3 is nilpotent in $R/3^k R$

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We have $R \cong R/2^k R \times R/3^k R$, we can now observe these rings more directlly. Take $R_2 = R/2^k R$ and $R_3 = R/3^k R$

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So we conclude that $R/N(R)$ is a subdirect product of copies of $\mathbb{Z}_2$ and $\mathbb{Z}_3$

Thank you to the NCUWM organizers, and to my advisor, Alex Diesl

Thank you for attending!

Questions?