# Almost All Wreath Product Character Values are Divisible by Given Primes

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MathILy-EST
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# Table of Contents

# The Symmetric Group

The **symmetric group** (a.k.a $\mathbf{S_N}$): all permutations of $N$ objects

Example: $\mathbf{N = 3}$

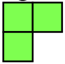
$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim (\mathbf{123}) \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \sim (\mathbf{132}) \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim (\mathbf{12})(\mathbf{3})$$

| | | |
|---|---|---|
| $\mathbf{e_1}: \quad 1 \to 2$ | $\mathbf{e_1}: \quad 1 \to 3$ | $\mathbf{e_1}: \quad 1 \to 2$ |
| $\mathbf{e_2}: \quad 2 \to 3$ | $\mathbf{e_2}: \quad 2 \to 1$ | $\mathbf{e_2}: \quad 2 \to 1$ |
| $\mathbf{e_3}: \quad 3 \to 1$ | $\mathbf{e_3}: \quad 3 \to 2$ | $\mathbf{e_3}: \quad 3 \to 3$ |

In the symmetric group, we are looking at **cycle types**.

Example: $\mathbf{N} = \mathbf{3}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow (12)(3) \rightsquigarrow \text{cycle type: } (21) \rightsquigarrow \text{Young diagram:}$$



Here are some elements in $S_3$ and their corresponding cycle types:

$$(12)(3) \to [21] \qquad (13)(2) \to [21]$$
$$(1)(2)(3) \to [1^3] \qquad (123) \to [3]$$

# Character Tables

A **character table** is an array of numbers encoding useful information about the representation theory of a finite group in compact form.

For $S_N$, columns and rows are indexed by **partitions of N**.

- **Column labels:** cycle types
- **Row labels:** irreducible representations

| $\boldsymbol{S_3}$ | $[1^3]$ | $[21]$ | $[3]$ |
|---|---|---|---|
| $\yng(3)$ | 1 | 1 | 1 |
| $\yng(2,1)$ | 2 | 0 | -1 |
| $\yng(1,1,1)$ | 1 | -1 | 1 |

An entry in the table is called a **character value**.

# Calculating Character Values

Use the combinatorial Murnaghan-Nakayama rule, which involves the idea of a **rimhook**.

Rules for finding valid rimhooks:

A **rimhook** is a connected subset of the Young diagram such that:

1. No outside boxes south or east
2. No box southeast of it (must be on the **borderstrip**)

NO NO NO YES

To calculate a character value, we **remove rimhooks** of lengths according to the *cycle types* (column labels) from the *characters* (row labels).

# Zeros of the Character Table

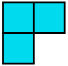
We are focused on the **zeros**!

Note: when we take the entire character table modulo a prime number,

$$0 \pmod{p} = \textbf{divisible by p}$$

**Forced zeros:** when you can't take any valid rimhook from the cycle type

- Example in $S_3$: try to take (21) from



It's impossible!

For sufficiently large N, we are (nearly) **guaranteed** an entire column of **forced zeros** when our column has a big first digit.

# Wreath Products

A **wreath product ($\wr$)** is a special combination of two groups. Here, we will be "wreathing" a finite group G with $S_N$.

| | | |
|---|---|---|
| $S_N$ | symmetric group | set of $N \times N$ permutation matrices |
| $G \wr S_N$ | wreath product of $G$ with $S_N$ | set of $N \times N$ permutation matrices with non-zero entries in $G$ |

Example: $\mathbf{B_N}$ (a.k.a the **signed symmetric group**!) where $\{-1, 1\} \in G$

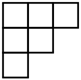
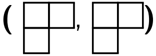
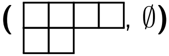
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in S_3 \qquad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \in B_3$$

Coin analogy!

# Row & Column Labels

Similar to symmetric group character tables!

We index rows and columns with **multi-partitions** of **cycle types** (instead of partitions).

| | |
|---|---|
|  | partition of 6 |
|  | 2-multi-partition of 6 |
|  | 3-multi-partition of 6 |
|  | 2-multipartition of 6 |

# Wreath Product Character Table

## Character Table of $B_2$

| | $[1^2, \emptyset]$ | $[2, \emptyset]$ | $[1, 1]$ | $[\emptyset, 1^2]$ | $[\emptyset, 2]$ |
|---|---|---|---|---|---|
| $\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \emptyset\right)$ | 1 | -1 | 1 | 1 | -1 |
| $(\square\square, \emptyset)$ | 1 | 1 | 1 | 1 | 1 |
| $(\square, \square)$ | 2 | 0 | 0 | -2 | 0 |
| $\left(\emptyset, \begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)$ | 1 | -1 | -1 | 1 | 1 |
| $(\emptyset, \square\square)$ | 1 | 1 | -1 | 1 | -1 |

# Previous Research

Recall from earlier: For sufficiently large N, we are (nearly) **guaranteed** an entire column of **forced zeros** when our column has a big first digit.

In 2020, **Sarah Peluse** and **Kannan Soundararajan** proved:

> "In the character table of $S_N$, for all primes $p << N$,
> the proportion of entries divisible by $p$ tends to 1 as $N \to \infty$."

Paraphrased: **Almost all symmetric group character values are divisible by given primes!**

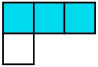
# Column Congruence mod p Lemma

For a prime $p << N$, Peluse and Soundararajan showed the following:

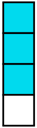
Entries in two columns of a symmetric group character table are **congruent mod p** if you can transform one Young diagram into another by either:

1. **cutting up** one big row into p equal rows
2. **mashing** p equal rows into one big row

Example in $S_4$ where $p = 3$ :



$\sim_3$

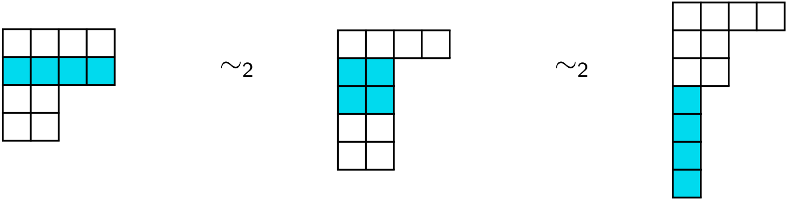


$(31) \sim_3 (1^4)$

# More transformations

This can be done repeatedly!

Example in $S_{12}$ for $\mathbf{p = 2}$:




$(4^2 2^2) \sim_2 (42^4) \sim_2 (42^2 1^4)$

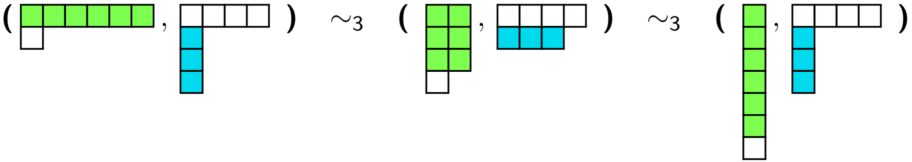
Reading **left to right**, we **cut up** a big row into 2 equal rows

Reading **right to left**, we **mash** 2 equal rows into a big row

# Wreath Product Lemma

Same idea as before (lots of cutting and mashing), but now we're dealing with multi-partitions, so we add in the **traffic rule**! (a.k.a. stay in your lane)

Example in $B_{14}$ where $p = 3$:




$(61\ ,\ 41^3) \quad \sim_3 \quad (2^31\ ,\ 43) \quad \sim_3 \quad (1^7\ ,\ 41^3)$

Lots of columns are congruent mod primes!

# Reasoning

Why do we want to mash and cut column labels to determine column congruence mod primes?

Because

1. Mashing gives us a **big first row**, which gives us...
2. A **forced zero**, which gives us...
3. $0 \pmod{p}$, which means **divisible by our given prime!**

From here, we do a bit more combinatorics to prove:

> "For wreath products of a finite group $G$ and $S_N$,
> the proportion of entries divisible by $p$ tends to 1 as $N \to \infty$."

**Almost All Wreath Product Character Values are Divisible by Given Primes!**

# Thank you!